SCHOOL ON CONFORMAL BLOCKS
ICMAT 15 - 19 OCTOBER 2012
ABSTRACTS AND REFERENCES

J. Andersen: [AU1], [AU2], [AU3], [AU4]

Isomorphism between the Combinatorial and the Conformal Field Theory Construction of Topological Quantum Field Theories

P. Belkale: [B1], [BM], [SV], [FGK]

Conformal blocks and hypergeometric local systems.

Conformal blocks give projective local systems on moduli spaces of curves with marked points. One can ask if they are realizable in “geometry”, i.e., as local subsystems of suitable Gauss-Manin local systems of cohomology of families of smooth projective varieties. We will discuss (in genus 0) the proof of Gawedzki et al’s conjecture that Schechtman-Varchenko forms are square integrable (this was proved first for sl(2) by Ramadas). Together with the flatness results of Schechtman-Varchenko, and the work of Ramadas, one obtains the desired realization and a flat unitary metric on conformal blocks.

The second topic is on characterizing the image of conformal blocks in the corresponding Gauss-Manin systems (joint work with S. Mukhopadhyay; the case of sl(2) is due to Looijenga and Varchenko).

Roughly the plan is to state the results and their context in the first lecture, to (sketch) the proof of the extension to compactifications (square integrability) result in the second, and to prove the “characterization of the image” in the third lecture.

N. Fakhruddin: [F1]

Chern classes of conformal blocks.

The theory of conformal blocks associates to a simple Lie algebra $\mathfrak{g}$, an integer ℓ called the level, and a collection of n irreducible representations of $\mathfrak{g}$ of level ℓ, vector bundles on the moduli stacks of stable curves of genus g with n marked points. When $g = 0$, these bundles are generated by global sections so their Chern classes give rise to nef cycles on the moduli stacks. In the first lecture I will describe a formula for the Chern classes in this case, focussing mainly on the first Chern class, and will discuss in more detail the case $\mathfrak{g} = sl_2$. In the second lecture I will consider the case of general simple Lie algebras but with $\ell = 1$. In the last lecture, I will explain the proof of the formula when (g, n) is $(0, 4)$ or $(1, 1)$; this suffices to compute the first Chern class for all (g, n).

N. Giansiracusa: [G1], [G2], [BG]

Geometric manifestations of conformal blocks line bundles.

In the first lecture I will provide some background on the moduli space of curves and on geometric invariant theory (GIT), providing the framework for the subsequent discussion of conformal blocks bundles. The main goals are to familiarize the audience with some basic properties of, and constructions relating to, the moduli of curves—especially in genus zero—and to mention some of the current open questions that conformal blocks vector bundles tie into.

In the next lecture, I'll describe a couple geometric constructions related to configurations of points on the line which yield a certain class of conformal blocks line bundles. This is the subject of a series of papers by myself and others (Bolognesi, Fakhruddin, Fedorchuk, Gibney) which was made possible through Fakhruddin's beautiful work on Chern classes of conformal blocks bundles.

In the third lecture I will present some other work (Alexeev, Gibney, Swinarski and their collaborators at University of Georgia) concerning conformal blocks from the perspective of divisor-theory on the moduli of curves. I'll conclude by mentioning some open questions in the subject and possible areas for future research.

E. Looijenga: [Lo1], [TUY]

(1) Statement of the main results on WZW models of Conformal Blocks [TUY]
Existence of a flat connection on the Verlinde bundle over (a $\mathbf{C}^*$-bundle over) the moduli $\mathcal{M}_{g,n}$ parametrizing smooth n-pointed genus-g curves, the propagation principle, the factorization property.
Virasoro algebra, Fock representation and its version for a symplectic local system.

(2) Loop algebras, Sugawara construction, the flat connection.

(3) Sketch of proof of propagation principle and factorization property. Genus zero case and the KZ-system.

C. Manon: [M1], [M2]

(1) The factorization rules and flat degeneration.

Toric degenerations of spaces are a way to replace geometric or algebraic questions with questions about polyhedral geometry. In this talk we discuss how the factorization rules for conformal blocks over $\bar{\mathcal{M}}_{g,n}$ can be used to construct flat degenerations of the Cox ring of the moduli of quasi-parabolic principal bundles on an n-marked curve of genus g and we give examples of when these degenerations are toric, to be discussed further in subsequent talks.

(2) Polytopes of $\mathrm{SL}_2(\mathbf{C})$ *conformal blocks.*

In this talk we discuss the affine semigroup algebras which result from degenerating the Cox ring of the moduli of quasi-parabolic $\mathrm{SL}_2(\mathbf{C})$-principal bundles on a marked curve. Starting with work of Sturmfels and Xu, we describe how structural features of these algebras can be obtained from studying the associated polyhedral geometry. We then show that the square of any effective line bundle on this moduli stack has a Koszul projective coordinate ring. Finally, we discuss the relationship between the moduli of quasi-parabolic $\mathrm{SL}_2(\mathbf{C})$-principal bundles and phylogenetic statistical models from mathematical biology revealed by the combinatorics.

(3) Oriented graphs and $\mathrm{SL}_3(\mathbf{C})$ *conformal blocks.*

In this talk we discuss how to take the degeneration constructed in the first talk one step further. We construct toric degenerations of the Cox ring of the moduli of quasi-parabolic $SL_3(\mathbf{C})$-principal bundles, and describe the resulting combinatorial objects. Along the way we will discuss further the relationship between conformal blocks and the space of invariant vectors in a tensor product of irreducible representations.

T.R. Ramadas: [Fa], [La], [H], [R1], [Ga], [TUY]

(1) Introduction to moduli spaces of vector bundles from different points of view: algebro-geometric, gauge theoretical and loop groups.

(2) Definition of Theta/Verlinde bundle and identification with conformal blocks. Hitchin's contruction of the flat connection. Laszlo's identification theorem of Hitchin's and WZW connection.

(3) Unitarity of the WZW connection in genus zero and rank 2. Relation with Gawedski's work.

References

[AU1] J.E. Andersen, K. Ueno: Geometric construction of modular functors from conformal field theory, Journal of Knot theory and its Ramifications. 16 2 (2007), 127-202.

[AU2] J.E. Andersen, K. Ueno: Abelian Conformal Field theories and Determinant Bundles, International Journal of Mathematics, Vol. 18, No. 8 (2007) 919-993.

[AU3] J.E. Andersen, K. Ueno: Modular functors are determined by their genus zero data, Journal of Quantum Topology, Vol. 3, Issue 3/4 (2012) 255291.

[AU4] J.E. Andersen, K. Ueno: Construction of the Reshetikhin-Turaev TQFT from conformal field theory. arXiv:1110.5027

[B1] P. Belkale: Unitarity of the KZ/Hitchin connection on conformal blocks in genus 0 for arbitrary Lie algebras, Journal de Mathématiques Pures et Appliquées, to appear, arXiv:1101.5846

[BM] P. Belkale, S. Mukhopadhyay: Conformal blocks and cohomology in genus 0, arXiv:1205.3199

[BG] M. Bologensi, N. Giansiracusa: Factorization of point configurations, cyclic covers and conformal blocks, arXiv:1208.4019

[F1] N. Fakhruddin: Chern classes of conformal blocks, arXiv:0904.2918

[FGK] F. Falceto, K. Gawedzki, A. Kupiainen: Scalar product of current blocks in WZW theory, Phys. Lett. B, Vol. 260, (1991), no. 1-2, 101108.

[Fa] G. Faltings: A proof for the Verlinde formula, J. Algebraic Geom. 3 (1994), no. 2, 347-374.

[H] N.J. Hitchin: Flat connections and geometric quantization, Comm. Math. Phys. 131 (1990), 347-380.

[Ga] K. Gawedzki, Lectures on conformal field theory, in Quantum Fields and Strings: a Course for Mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), 727805, Amer. Math. Soc., Providence, RI, 1999.

[G1] N. Giansiracusa: Conformal blocks and rational normal curves, arXiv:1012.4835

[G2] N. Giansiracusa, A. Gibney: The cone of type A, level one conformal blocks divisors, Advances in Mathematics 231 (2012), 798-814.

[La] Y. Laszlo: Hitchin' s and WZW connections are the same, J. Differential Geom. 49 (1998), 547-576.

[Lo1] E. Looijenga: From WZW models to modular functors, Handbook of Moduli, arxiv:1009.2245

[M1] C. Manon: The Algebra of Conformal Blocks, arXiv:0910.0577

[M2] C. Manon: Coordinate rings for the moduli of $SL_2(\mathbf{C})$ quasi-parabolic principal bundles on a curve and toric fiber products, arXiv:1105.2045

[R1] T.R. Ramadas: The "Harder-Narasimhan trace" and unitarity of the KZ/Hitchin connection: genus 0, Annals of Mathematics, 169 (2009), 139

[SV] V. V. Schechtman, A. N. Varchenko: Arrangements of hyperplanes and Lie algebra homology, Invent. Math., Vol. 106, 139194 (1991)

[TUY] A. Tsuchiya, K. Ueno, Y. Yamada: Conformal Field Theory on Universal Family of Stable Curves with Gauge Symmetries, Advanced Studies in Pure Mathematics 19 (1989), Kinokuniya Shoten and Academic Press, 459-566