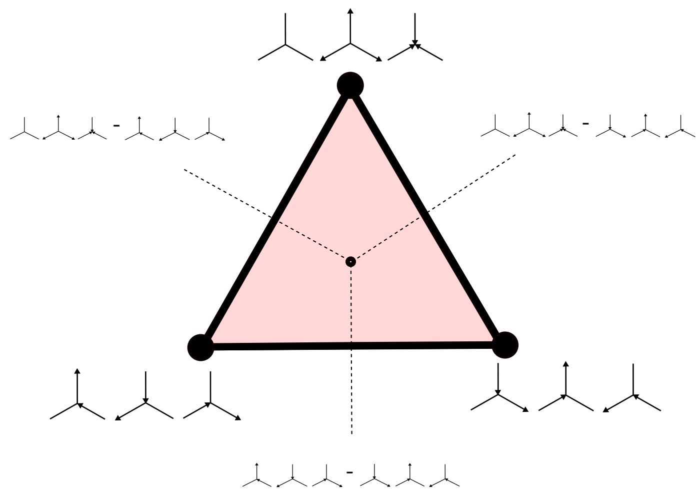
Combinatorial commutative algebra of conformal blocks



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Outline of Talks

Talk 1: Review combinatorics and commutative algebra attached to conformal blocks, introduce flat degenerations of coordinate rings of moduli of principal bundles.

Talk 2: The $SL_2(\mathbb{C})$ case; structure of conformal blocks polytopes; combinatorial commutative algebra of $SL_2(\mathbb{C})$ conformal blocks.

Talk 3: The $SL_3(\mathbb{C})$ case; combinatorics of tensors; relationship between conformal blocks and mathematical biology.

Review from talks I, II

-$\mathfrak{g}$: a simple Lie algebra over $\mathbb{C}$.

-G: A simple algebraic group over $\mathbb{C}$ with $Lie(G) = \mathfrak{g}$.

-Δ: A Weyl chamber of $\mathfrak{g}$.

-$B \subset G$: A Borel subgroup.

-Δ_L: The $L-$restricted Weyl chamber of $\mathfrak{g}$.

Review from talks I,II

- $(C, \vec{p}) \in \bar{\mathcal{M}}_{g,n}$: a stable, $n-$marked curve of genus g.

- $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$: an $n-$tuple of $sl_3(\mathbb{C})$ dominant weights, $\lambda_i \in \mathbb{Z}^2_{\geq 0}$.

- $V_{C,\vec{p}}(\vec{\lambda}, L)$: the space of $sl_3(\mathbb{C})$ conformal blocks on $(C, \vec{p})$ with weight data $(\vec{\lambda}, L)$

- $\mathbb{V}_{g,n}(\vec{\lambda}, L) = dim[V_{C,\vec{p}}(\vec{\lambda}, L)]$

Review from talks I,II

-$\mathcal{M}_{C,\vec{p}}(SL_3(\mathbb{C}))$: the moduli of quasi-parabolic $SL_3(\mathbb{C})$ principal bundles on $(C,\vec{p})$.

-$\mathcal{L}(\vec{\lambda},L)$: The line bundle on $\mathcal{M}_{C,\vec{p}}(SL_3(\mathbb{C}))$ with weight data $(\vec{\lambda},L)$.

-$V_{C,\vec{p}}(SL_3(\mathbb{C})) = Cox(\mathcal{M}_{C,\vec{p}}(SL_3(\mathbb{C})))$.

$$H^0(\mathcal{M}_{C,\vec{p}}(SL_3(\mathbb{C})),\mathcal{L}(\vec{\lambda},L)) = V_{C,\vec{p}}(\vec{\lambda},L)$$

Review from talks I,II

$$V_{C,\vec{p}}(SL_3(\mathbb{C})) = \bigoplus_{\vec{\lambda},L} V_{C,\vec{p}}(\vec{\lambda}, L)$$

$$R_{C,\vec{p}}(\vec{r}, L) = \bigoplus_{N \geq 0} V_{C,\vec{p}}(N\vec{\lambda}, NL)$$

Review from talks I, II

Questions:

-What generates the conformal blocks?

-What relations hold among these generators?

-How do we count the conformal blocks?

Review from talks I,II

Questions:

-What generates $V_{C,\vec{p}}(SL_3(\mathbb{C}))$?

-What relations hold among these generators?

-What is the multigraded Hilbert function of $V_{C,\vec{p}}(SL_3(\mathbb{C}))$?

Review from talks I,II

For every trivalent graph with first Betti number g and n leaves, there is a flat degeneration

$$V_{C,\vec{p}}(G) \Rightarrow [\bigotimes_{v \in V(\Gamma)} V_{0,3}(G)]^{T_\Gamma}$$

Review from talks I,II

What went right for $SL_2(\mathbb{C})$?

Proposition [Quantum Clebsch-Gordon rule] :

The dimension of $V_{0,3}(r_1, r_2, r_3, L)$ is either 1 or 0. It is dimension 1 if and only if $r_1 + r_2 + r_3$ is even, $\leq 2L$, and r_1, r_2, r_3 are the side-lengths of a triangle

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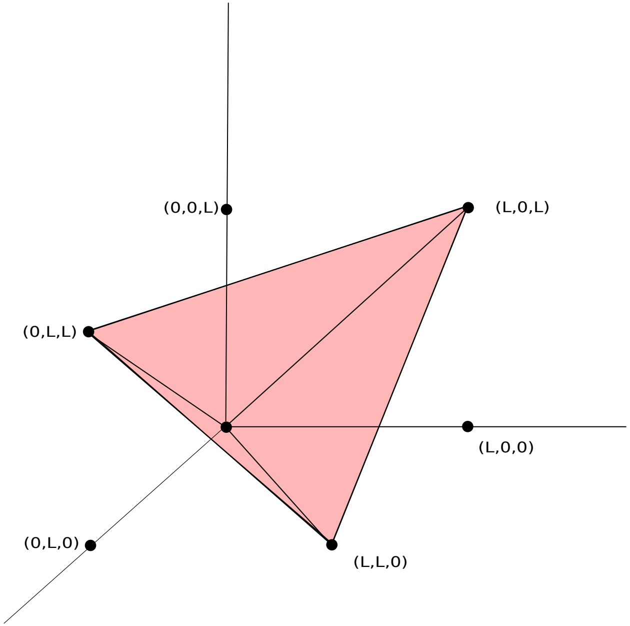
Review from talks I,II

What went right for $SL_2(\mathbb{C})$?

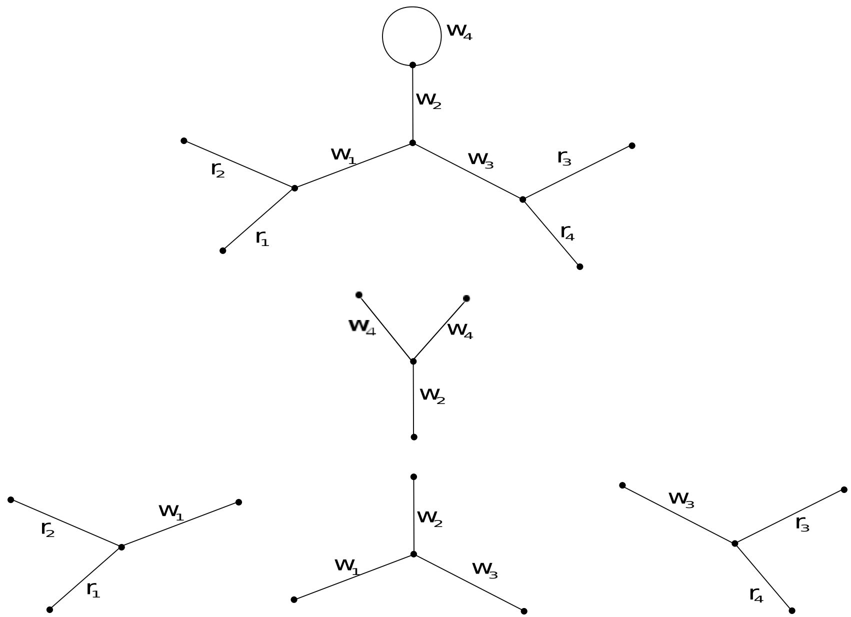


$\frac{1}{2}$

$$V_{0,3}(SL_2(\mathbb{C})) = \mathbb{C}[P_3(1)]$$

Review from talks I,II

What went right for $SL_2(\mathbb{C})$?



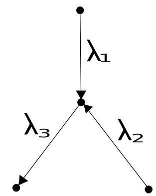
Allows us to build a toric degeneration of $V_{C,\vec{p}}(SL_2(\mathbb{C}))$ out of copies of $V_{0,3}(SL_2(\mathbb{C}))$.

The algebra $V_{0,3}(G)$

In general it would suffice to have a toric degeneration of $V_{0,3}(G)$,

$$V_{0,3}(G) \Rightarrow \mathbb{C}[P_3],$$

which respects the multigrading by $\mathcal{X}(B)^3 \times \mathbb{Z}$.



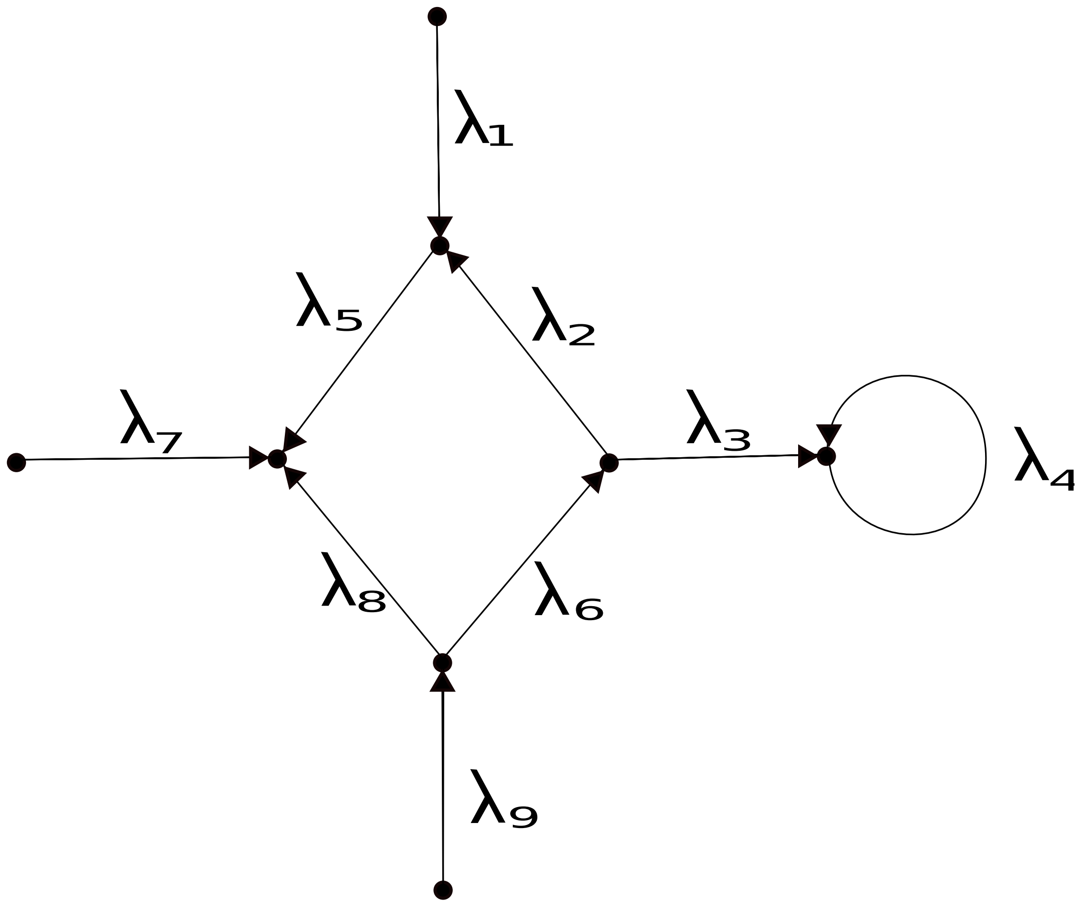
The algebra $V_{0,3}(G)$

$$V_{C,\vec{p}}(G) \Rightarrow [\bigotimes_{v \in V(\Gamma)} V_{0,3}(G)]^{T_\Gamma} \Rightarrow [\bigotimes_{v \in V(\Gamma)} \mathbb{C}[P_3]]^{T_\Gamma}$$

$$[\bigotimes_{v \in V(\Gamma)} \mathbb{C}[P_3]]^{T_\Gamma} = \mathbb{C}[P_\Gamma]$$

Where P_Γ is the fiber product of $|V(\Gamma)|$ copies of P_3 over copies of Δ_L.

The algebra $V_{0,3}(G)$



The algebra $V_{0,3}(G)$

[Tsuchiya, Ueno, Yamada]: The space $V_{0,3}(\lambda, \eta, \mu, L)$ can be identified with a subspace of $[V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^{\mathfrak{g}}$

Tensors and Conformal Blocks

Let $\theta(sl_2(\mathbb{C})) \subset \mathfrak{g}$ be the copy of $sl_2(\mathbb{C})$ corresponding to the longest root. We branch each $V(\lambda)$ along this sub-algebra.

$$V(\lambda^*) = \bigoplus_i W_{\lambda,i}$$

Let $W_L(\lambda,\eta,\mu) = \bigoplus_{i+j+k\leq 2L} W_{\lambda,i} \otimes W_{\eta,j} \otimes W_{\mu,k}$.

$$V_{0,3}(\lambda,\eta,\mu,L) = W_L(\lambda,\eta,\mu) \cap [V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^{\mathfrak{g}}$$

Tensors and Conformal Blocks

Invariant tensors have many combinatorial descriptions, including polyhedral counting rules.

[Berenstein, Zelevinsky]: There is a polytope $P_{\vec{i},3}(\lambda, \eta, \mu)$ with integral points in bijection with a basis of the space $[V(\lambda) \otimes V(\eta) \otimes V(\mu)]^{\mathfrak{g}}$

[Howe, Lee]: There is a polytope $L(\lambda, \eta, \mu)$ with integral points in bijection with a basis of the space $[V(\lambda) \otimes V(\eta) \otimes V(\mu)]^{sl_m(\mathbb{C})}$

Tensors and Conformal Blocks

[Zhelobenko]: The space $[V(\lambda) \otimes V(\eta) \otimes V(\mu)]^{\mathfrak{g}}$ can be identified with a subspace $V_{\mu,\lambda-\mu}(\eta) \subset V(\eta)$

This subspace inherits the dual canonical/crystal basis of Kashiwara/Lusztig. For every reduced decomposition $\vec{i}$ of the longest element w_0 of the Weyl group of $\mathfrak{g}$, this basis is labelled by corresponding integer "string parameters." These parameters give the lattice points of $P_{\vec{i},3}(\lambda, \eta, \mu)$.

The depth rule of Gepner,Witten

$$V_{0,3}(\lambda, \eta, \mu, L) \subset [V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^{\mathfrak{g}} \subset V_{\mu^*, \eta^* - \mu^*}(\lambda)$$

The conformal blocks subspace can be described as those vectors in $V_{\mu^*, \eta^* - \mu^*}(\lambda)$ which vanish under the action of $e_\theta^{L - \mu^*(H_\theta) + 1}$, where e_θ is the raising operator corresponding to the longest root.

The algebra $R_3(G)$

Define $R_G = \bigoplus_{\lambda \in \Delta} V(\lambda)$, with multiplication

$$V(\lambda) \otimes V(\eta) \to V(\lambda + \eta)$$

This is the Cox ring of the full flag variety G/B, and the coordinate ring of the affine variety G/U, where $U \subset G$ is maximal unipotent.

The algebra $R_3(G)$

Define $R_3(G) = [R_G \otimes R_G \otimes R_G]^G$.

$$R_3(G) = [R_G \otimes R_G \otimes R_G]^{\mathfrak{g}} = \bigoplus_{\lambda,\eta,\mu \in \Delta} [V(\lambda) \otimes V(\eta) \otimes V(\mu)]^{\mathfrak{g}}$$

The algebra $R_3(G)$

[M]: For each $\vec{i}$ there is toric degeneration of $R_3(G)$ to an affine semigroup algebra $\mathbb{C}[P_{\vec{i},3}]$ which respects the multigrading by $\mathcal{X}(B)^3$.

Here $P_{\vec{i},3}$ is a polyhedral cone with a map $\pi_3 : P_{\vec{i},3} \to \Delta^3$. The fiber $\pi^{-1}(\lambda, \eta, \mu)$ is the polytope $P_{\vec{i},3}(\lambda, \eta, \mu)$.

The algebras $R_3(G)$ and $V_{0,3}(G)$

The subspaces $V_{0,3}(\lambda, \eta, \mu, L) \subset [V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^{\mathfrak{g}}$ are compatible with the multiplication operation in $R_3(G)$.

$$V_{0,3}(G) \subset R_3(G) \otimes \mathbb{C}[t]$$

$$V_{0,3}(\lambda, \eta, \mu, L) \subset [V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^{\mathfrak{g}} t^L$$

The algebras $R_3(G)$ and $V_{0,3}(G)$

The function which assigns an invariant tensor $T \in [V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^{\mathfrak{g}}$ the minimum level L such that $T \in V_{0,3}(\lambda, \eta, \mu, L)$ defines a valuation v_θ on the algebras $R_3(G)$ and $V_{0,3}(G)$.

Valuations:

-$v(ab) = v(a) + v(b)$

-$v(a + b) \leq max\{v(a), v(b)\}$

-$v(0) = -\infty$

-$v(C) = 0,\ C \in \mathbb{C}$.

The algebras $R_3(G)$ and $V_{0,3}(G)$

Idea: Port the combinatorial commutative algebra of $R_3(G)$ over to $V_{0,3}(G)$.

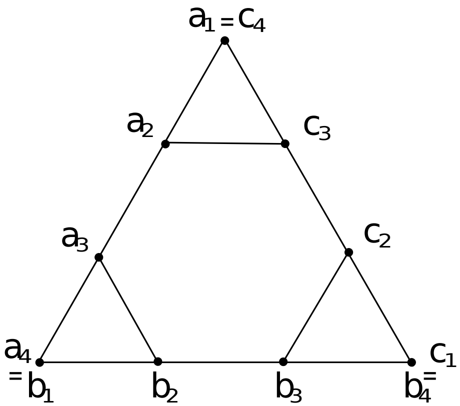
Idea: Find a "nice" (good?) basis of $[V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^{\mathfrak{g}}$ which restricts to a basis of each space $V_{0,3}(\lambda, \eta, \mu, L)$.

The algebra $R_3(SL_3(\mathbb{C}))$

BZ_3: A "string cone" for $SL_3(\mathbb{C})$.

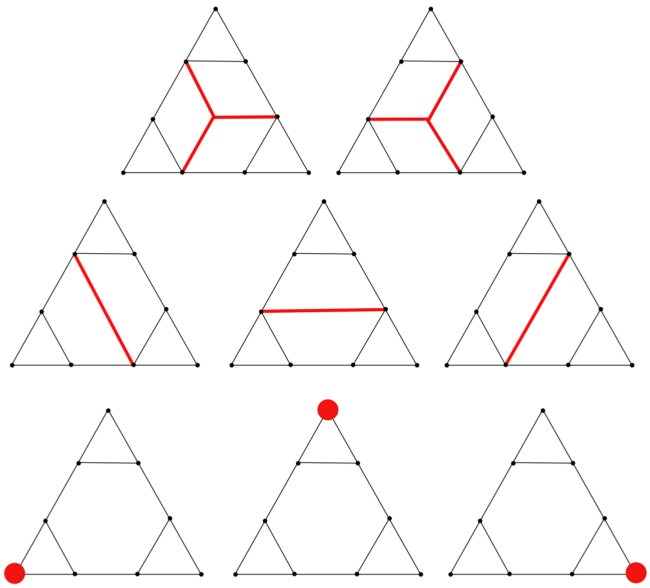
$[V(\lambda) \otimes V(\eta) \otimes V(\mu)]^{sl_3(\mathbb{C})}$

- all ≥ 0 integer entries,
- $(a_1 + a_2, a_3 + a_3) = \lambda$, $(b_1 + b_2, b_3 + b_4) = \eta$, $(c_1 + c_2, c_3 + c_4) = \mu$,
- $a_2 + a_3 = c_2 + b_3$, $b_2 + b_3 = a_2 + c_3$, $c_2 + c_3 = b_2 + a_3$.



The algebra $R_3(SL_3(\mathbb{C}))$

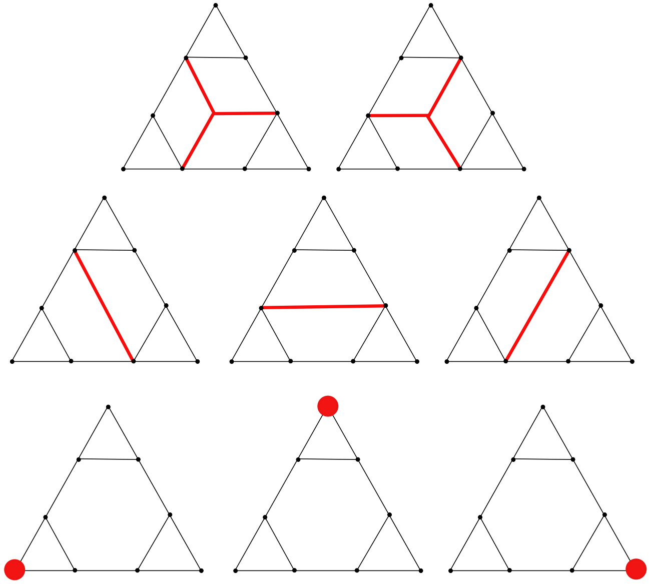
The semigroup algebra $\mathbb{C}[BZ_3]$ is generated by 8 elements.



$$X, Y, P_{12}, P_{23}, P_{31}, P_{21}, P_{32}, P_{13}$$

The algebra $R_3(SL_3(\mathbb{C}))$

Subject to one relation.



$$XY = P_{12}P_{23}P_{31}$$

The algebra $R_3(SL_3(\mathbb{C}))$

We can lift this information to a presentation of $R_3(SL_3(\mathbb{C}))$.

$$R_3(SL_3(\mathbb{C})) =$$

$$\mathbb{C}[X, Y, P_{12}, P_{23}, P_{31}, P_{21}, P_{32}, P_{13}] / < XY - P_{12}P_{23}P_{31} + P_{21}P_{32}P_{13} >$$

The algebra $R_3(SL_3(\mathbb{C}))$

$V(\omega_1) = \mathbb{C}^3$

$V(\omega_1^*) = \bigwedge^2(\mathbb{C}^3)$

Here P_{ij} is the invariant bilinear form in $V(\omega_1) \otimes V(\omega_1^*)$, where ω_1 is in the $i-$th place and ω_1^* is in the $j-$th place.

X and Y are the determinant invariants in $V(\omega_1)^{\otimes 3}$ and $V(\omega_1^*)^{\otimes 3}$, respectively.

The algebra $V_{0,3}(SL_3(\mathbb{C}))$

Can we use these tensors to get a basis of $V_{0,3}(\lambda, \eta, \mu, L)$?

Each generator $X, Y, P_{12}, P_{23}, P_{31}, P_{21}, P_{32}, P_{13}$ has v_θ value 1

The algebra $V_{0,3}(SL_3(\mathbb{C}))$

Can we use these tensors to get a basis of $V_{0,3}(\lambda, \eta, \mu, L)$?

There is a basis of $[V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^{sl_3(\mathbb{C})}$ by monomials in the generators X, Y, P_{12}, P_{23}, P_{31}, P_{21}, P_{32}, P_{13} each member of which has a distinct v_θ value.

This basis restricts to a basis of $V_{0,3}(\lambda, \eta, \mu, L)$.

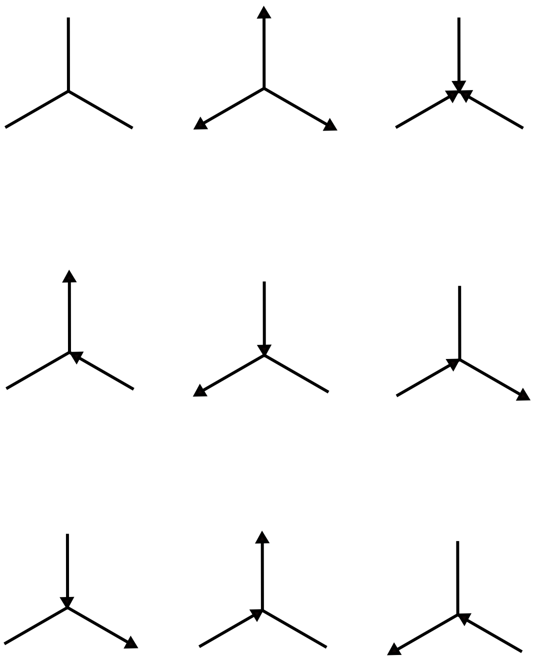
The algebra $V_{0,3}(SL_3(\mathbb{C}))$

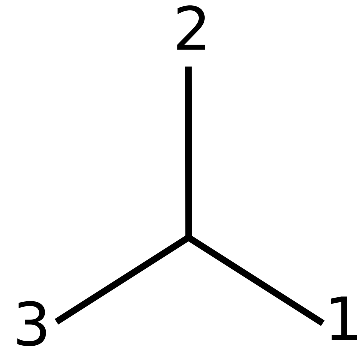
This can be used to construct a presentation of $V_{0,3}(SL_3(\mathbb{C}))$.

$$V_{0,3}(SL_3(\mathbb{C})) =$$

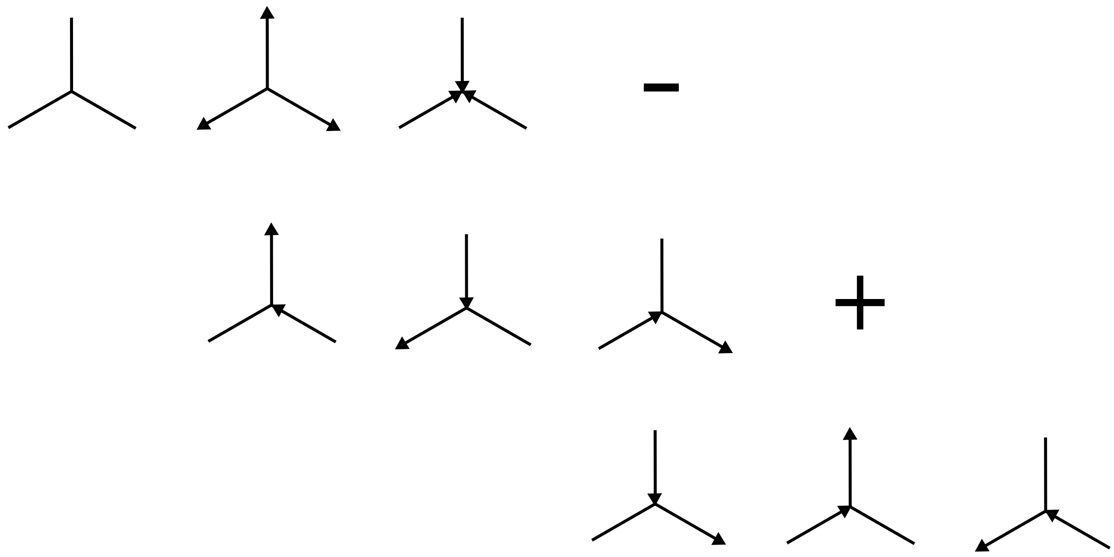
$$\mathbb{C}[Z, X, Y, P_{12}, P_{23}, P_{31}, P_{21}, P_{32}, P_{13}]/ < ZXY - P_{12}P_{23}P_{31} + P_{21}P_{32}P_{13} >$$

The algebra $V_{0,3}(SL_3(\mathbb{C}))$





The algebra $V_{0,3}(SL_3(\mathbb{C}))$



Toric degenerations of $V_{0,3}(SL_3(\mathbb{C}))$

$$ZXY - P_{12}P_{23}P_{31}$$

$$ZXY + P_{21}P_{32}P_{13}$$

$$P_{21}P_{32}P_{13} - P_{12}P_{23}P_{31}$$

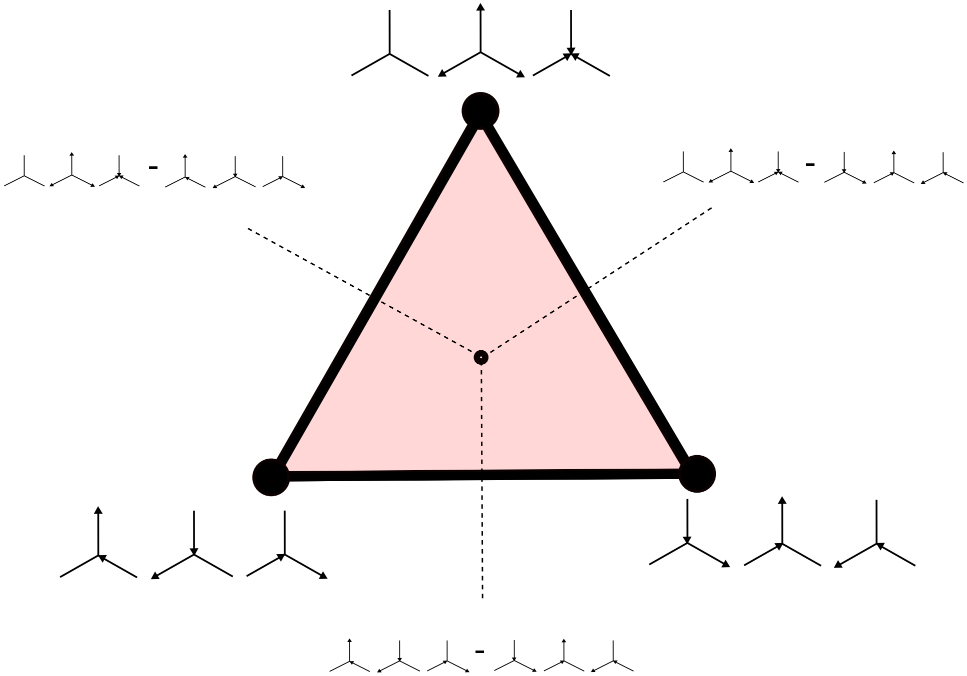
Polyhedral counting rules: $SL_3(\mathbb{C})$

$$\mathbb{V}_{0,3}(\lambda, \mu, \eta, L) = L - max\{\lambda_1 + \lambda_2, \mu_1 + \mu_2, \eta_1 + \eta_2, L_1, L_2\} + 1$$

$$L_1 = \frac{1}{3}(2(\lambda_1 + \mu_1 + \eta_1) + \lambda_2 + \mu_2 + \eta_2) - min\{\lambda_1, \mu_1, \eta_1\}$$

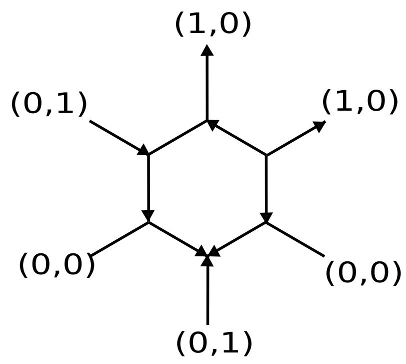
$$L_2 = \frac{1}{3}(2(\lambda_2 + \mu_2 + \eta_2) + \lambda_1 + \mu_1 + \eta_1) - min\{\lambda_2, \mu_2, \eta_2\}.$$

The multigraded Hilbert scheme for $\mathbb{V}_{0,3}$



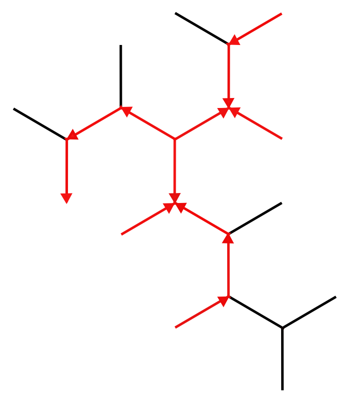
Toric degenerations of $V_{C,\vec{p}}(SL_3(\mathbb{C}))$

For each trivalent graph Γ, there are $3^{|V(\Gamma)|}$ toric degenerations of $V_{C,\vec{p}}(SL_3(\mathbb{C}))$.



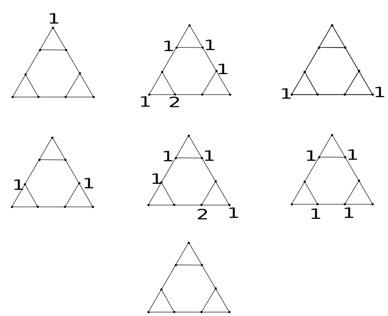
Commutative algebra for genus 0

For $(\mathbb{P}^1, \vec{p})$ generic, the algebra $V_{\mathbb{P}^1,\vec{p}}(SL_3(\mathbb{C}))$ is generated by the 3^{n-1} conformal blocks of level 1, and has relations generated by homogenous forms of degree $2, 3$.



Commutative algebra for genus 1

For $(C, \vec{p})$ generic with $g = 1$, the algebra $V_{C,\vec{p}}(SL_3(\mathbb{C}))$ is generated by conformal blocks of level $1, 2, 3$.



Possible elements of a generalization

-The tensor product invariants of Howe, Lee; Berenstein, Zelevinsky.

-valuations suggest a role for tropical geometry

Some mathematical biology

-Sturmfels, Sullivant, Toric ideals of phylogenetic invariants, Journal of Computational Biology 12 (2005) 204-228.

-Phylogenetic variety: algebraic variety cut out by binomial equations which vanish on marginal probabilities from a phylogenetic statistical model.

-these are tools for reconstructing ancestral relationships

Some mathematical biology

-Given an oriented, rooted graph Γ, each vertex $v \in V(\Gamma)$ receives a (possibly $k > 0$ ary) random variable.

-Put probability distribution at the root vertex.

-each edge $e \in E(\Gamma)$ has a transition matrix $A(e)$, used to recursively compute a distribution at each subsequent vertex.

-The resulting marginal probabilities at the leaves are forced to satisfy equations determined by the matrices $A(e)$.

Some mathematical biology

One source of matrices $A(v)$ are finite Abelian groups G. For each G there is a phylogenetic "group based" model which determines a binomial ideal $I_{G,\mathcal{T}}$ for every structure tree $\mathcal{T}$.

For $G = \mathbb{Z}/2\mathbb{Z}$, $I_{\mathcal{T},\mathbb{Z}/2\mathbb{Z}}$ is the ideal defining $\mathbb{C}[P_{\mathcal{T}}(1)]$

Some mathematical biology

Let $A_{\mathcal{T},G}$ denote the corresponding affine semigroup algebra.

[Kubjas, M]: $A_{\mathcal{T},\mathbb{Z}/3\mathbb{Z}}$ is a natural quotient of $[\bigotimes_{v \in V(\mathcal{T})} V_{0,3}(SL_3(\mathbb{C}))]^{T_{\mathcal{T}}}$

$A_{\mathcal{T},\mathbb{Z}/m\mathbb{Z}}$ is a natural sub-quotient of $[\bigotimes_{v \in V(\mathcal{T})} V_{0,3}(SL_m(\mathbb{C}))]^{T_{\mathcal{T}}}$

Thankyou!