Combinatorial commutative algebra of conformal blocks

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Outline of Talks

Talk 1: Review combinatorics and commutative algebra attached to conformal blocks, introduce flat degenerations of coordinate rings of moduli of principal bundles.

Talk 2: The $SL_2(\mathbb{C})$ case; structure of conformal blocks polytopes; combinatorial commutative algebra of $SL_2(\mathbb{C})$ conformal blocks.

Talk 3: The $SL_3(\mathbb{C})$ case; combinatorics of tensors; relationship between conformal blocks and mathematical biology.
- $\mathfrak{g}$: a simple Lie algebra over $\mathbb{C}$.

- $G$: A simple algebraic group over $\mathbb{C}$ with $\text{Lie}(G) = \mathfrak{g}$.

- $\Delta$: A Weyl chamber of $\mathfrak{g}$.

- $B \subset G$: A Borel subgroup.

- $\Delta_L$: The $L$–restricted Weyl chamber of $\mathfrak{g}$. 
- $(C, \vec{p}) \in \tilde{\mathcal{M}}_{g,n}$: a stable, $n$–marked curve of genus $g$.

- $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$: an $n$–tuple of $sl_3(\mathbb{C})$ dominant weights, $\lambda_i \in \mathbb{Z}_{\geq 0}^2$.

- $V_{C,\vec{p}}(\vec{\lambda}, L)$: the space of $sl_3(\mathbb{C})$ conformal blocks on $(C, \vec{p})$ with weight data $(\vec{\lambda}, L)$

- $\mathbb{V}_{g,n}(\vec{\lambda}, L) = dim[V_{C,\vec{p}}(\vec{\lambda}, L)]$
Review from talks I,II

$\mathcal{M}_{C,\vec{p}}(SL_3(\mathbb{C}))$: the moduli of quasi-parabolic $SL_3(\mathbb{C})$ principal bundles on $(C, \vec{p})$.

$\mathcal{L}(\vec{\lambda}, L)$: The line bundle on $\mathcal{M}_{C,\vec{p}}(SL_3(\mathbb{C}))$ with weight data $(\vec{\lambda}, L)$.

$V_{C,\vec{p}}(SL_3(\mathbb{C})) = Cox(\mathcal{M}_{C,\vec{p}}(SL_3(\mathbb{C})))$.

$$H^0(\mathcal{M}_{C,\vec{p}}(SL_3(\mathbb{C})), \mathcal{L}(\vec{\lambda}, L)) = V_{C,\vec{p}}(\vec{\lambda}, L)$$
Review from talks I,II

\[ V_{C,\bar{p}}(SL_3(\mathbb{C})) = \bigoplus_{\lambda, L} V_{C,\bar{p}}(\lambda, L) \]

\[ R_{C,\bar{p}}(\vec{r}, L) = \bigoplus_{N \geq 0} V_{C,\bar{p}}(N\lambda, NL) \]
Questions:

- What generates the conformal blocks?

- What relations hold among these generators?

- How do we count the conformal blocks?
Review from talks I,II

Questions:

- What generates $V_{\mathbb{C}, \vec{p}}(SL_3(\mathbb{C}))$?

- What relations hold among these generators?

- What is the multigraded Hilbert function of $V_{\mathbb{C}, \vec{p}}(SL_3(\mathbb{C}))$?
For every trivalent graph with first Betti number $g$ and $n$ leaves, there is a flat degeneration

$$V_{C, \vec{p}}(G) \Rightarrow [ \bigotimes_{v \in V(\Gamma)} V_{0,3}(G) ]^{T\Gamma}$$
Review from talks I,II

What went right for $SL_2(\mathbb{C})$?

Proposition [Quantum Clebsch-Gordon rule] :
The dimension of $V_{0,3}(r_1, r_2, r_3, L)$ is either 1 or 0. It is dimension 1 if and only if $r_1 + r_2 + r_3$ is even, $\leq 2L$, and $r_1, r_2, r_3$ are the side-lengths of a triangle.
What went right for $SL_2(\mathbb{C})$?

$$V_{0,3}(SL_2(\mathbb{C})) = \mathbb{C}[P_3(1)]$$
What went right for $SL_2(\mathbb{C})$?

Allows us to build a toric degeneration of $V_{C,\bar{p}}(SL_2(\mathbb{C}))$ out of copies of $V_{0,3}(SL_2(\mathbb{C}))$. 
The algebra $V_{0,3}(G)$

In general it would suffice to have a toric degeneration of $V_{0,3}(G)$,

$$V_{0,3}(G) \Rightarrow \mathbb{C}[P_3],$$

which respects the multigrading by $\mathcal{X}(B)^3 \times \mathbb{Z}$. 

\[ \lambda_1 \]
\[ \lambda_2 \]
\[ \lambda_3 \]

\[ \lambda \]
The algebra $V_{0,3}(G)$

\[
V_{C,\vec{p}}(G) \Rightarrow \left[ \bigotimes_{v \in V(\Gamma)} V_{0,3}(G) \right]^{T\Gamma} \Rightarrow \left[ \bigotimes_{v \in V(\Gamma)} \mathbb{C}[P_3] \right]^{T\Gamma}
\]

\[
\left[ \bigotimes_{v \in V(\Gamma)} \mathbb{C}[P_3] \right]^{T\Gamma} = \mathbb{C}[P_\Gamma]
\]

Where $P_\Gamma$ is the fiber product of $|V(\Gamma)|$ copies of $P_3$ over copies of $\Delta_L$. 
The algebra $V_{0,3}(G)$
The algebra $V_{0,3}(G)$

[Tsuchiya, Ueno, Yamada]: The space $V_{0,3}(\lambda, \eta, \mu, L)$ can be identified with a subspace of $[V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^g$. 
Let $\theta(sl_2(\mathbb{C})) \subset g$ be the copy of $sl_2(\mathbb{C})$ corresponding to the longest root. We branch each $V(\lambda)$ along this sub-algebra.

$$V(\lambda^*) = \bigoplus W_{\lambda,i}$$

Let $W_L(\lambda, \eta, \mu) = \bigoplus_{i+j+k \leq 2L} W_{\lambda,i} \otimes W_{\eta,j} \otimes W_{\mu,k}$.

$$V_{0,3}(\lambda, \eta, \mu, L) = W_L(\lambda, \eta, \mu) \cap [V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^g$$
Tensors and Conformal Blocks

Invariant tensors have many combinatorial descriptions, including polyhedral counting rules.

[Berenstein, Zelevinsky]: There is a polytope $P_{i,3}(\lambda, \eta, \mu)$ with integral points in bijection with a basis of the space $[V(\lambda) \otimes V(\eta) \otimes V(\mu)]^g$

[Howe, Lee]: There is a polytope $L(\lambda, \eta, \mu)$ with integral points in bijection with a basis of the space $[V(\lambda) \otimes V(\eta) \otimes V(\mu)]^{slm(\mathbb{C})}$
Tensors and Conformal Blocks

[Zhelobenko]: The space \[ V(\lambda) \otimes V(\eta) \otimes V(\mu) \] can be identified with a subspace \( V_{\mu,\lambda-\mu}(\eta) \subset V(\eta) \)

This subspace inherits the dual canonical/crystal basis of Kashiwara/Lusztig. For every reduced decomposition \( \vec{i} \) of the longest element \( w_0 \) of the Weyl group of \( g \), this basis is labelled by corresponding integer "string parameters." These parameters give the lattice points of \( P_{i,3}(\lambda, \eta, \mu) \).
The depth rule of Gepner, Witten

\[ V_{0,3}(\lambda, \eta, \mu, L) \subset [V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^g \subset V_{\mu^*, \eta^*-\mu^*}(\lambda) \]

The conformal blocks subspace can be described as those vectors in \( V_{\mu^*, \eta^*-\mu^*}(\lambda) \) which vanish under the action of \( e_L^{-\mu^*(H_\theta)+1} \), where \( e_\theta \) is the raising operator corresponding to the longest root.
The algebra $R_3(G)$

Define $R_G = \bigoplus_{\lambda \in \Delta} V(\lambda)$, with multiplication

$$V(\lambda) \otimes V(\eta) \rightarrow V(\lambda + \eta)$$

This is the Cox ring of the full flag variety $G/B$, and the coordinate ring of the affine variety $G/U$, where $U \subset G$ is maximal unipotent.
The algebra $R_3(G)$

Define $R_3(G) = [R_G \otimes R_G \otimes R_G]^G$.

$$R_3(G) = [R_G \otimes R_G \otimes R_G]^g = \bigoplus_{\lambda, \eta, \mu \in \Delta} [V(\lambda) \otimes V(\eta) \otimes V(\mu)]^g$$
The algebra $R_3(G)$

[M]: For each $\vec{i}$ there is toric degeneration of $R_3(G)$ to an affine semigroup algebra $\mathbb{C}[P_{\vec{i},3}]$ which respects the multigrading by $\mathcal{X}(B)^3$.

Here $P_{\vec{i},3}$ is a polyhedral cone with a map $\pi_3 : P_{\vec{i},3} \to \Delta^3$. The fiber $\pi^{-1}(\lambda, \eta, \mu)$ is the polytope $P_{\vec{i},3}(\lambda, \eta, \mu)$. 
The algebras $R_3(G)$ and $V_{0,3}(G)$

The subspaces $V_{0,3}(\lambda, \eta, \mu, L) \subset [V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^g$ are compatible with the multiplication operation in $R_3(G)$.

$$V_{0,3}(G) \subset R_3(G) \otimes \mathbb{C}[t]$$

$$V_{0,3}(\lambda, \eta, \mu, L) \subset [V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^{g_t L}$$
The algebras $R_3(G)$ and $V_{0,3}(G)$

The function which assigns an invariant tensor $T \in [V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^g$ the minimum level $L$ such that $T \in V_{0,3}(\lambda, \eta, \mu, L)$ defines a valuation $v_\theta$ on the algebras $R_3(G)$ and $V_{0,3}(G)$.

Valuations:
- $v(ab) = v(a) + v(b)$
- $v(a + b) \leq \max\{v(a), v(b)\}$
- $v(0) = -\infty$
- $v(C) = 0, \ C \in \mathbb{C}$. 
The algebras $R_3(G)$ and $V_{0,3}(G)$

Idea: Port the combinatorial commutative algebra of $R_3(G)$ over to $V_{0,3}(G)$.

Idea: Find a "nice" (good?) basis of $[V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^g$ which restricts to a basis of each space $V_{0,3}(\lambda, \eta, \mu, L)$. 
The algebra $R_3(SL_3(\mathbb{C}))$

$BZ_3$: A "string cone" for $SL_3(\mathbb{C})$.

$[V(\lambda) \otimes V(\eta) \otimes V(\mu)]^{sl_3(\mathbb{C})}$

- all $\geq 0$ integer entries,
- $(a_1 + a_2, a_3 + a_3) = \lambda$, $(b_1 + b_2, b_3 + b_4) = \eta$, $(c_1 + c_2, c_3 + c_4) = \mu$,
- $a_2 + a_3 = c_2 + b_3$, $b_2 + b_3 = a_2 + c_3$, $c_2 + c_3 = b_2 + a_3$. 

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The algebra $R_3(SL_3(\mathbb{C}))$

The semigroup algebra $\mathbb{C}[BZ_3]$ is generated by 8 elements.

$X, Y, P_{12}, P_{23}, P_{31}, P_{21}, P_{32}, P_{13}$
The algebra $R_3(SL_3(\mathbb{C}))$

Subject to one relation.

\[ XY = P_{12}P_{23}P_{31} \]
The algebra $R_3(SL_3(\mathbb{C}))$

We can lift this information to a presentation of $R_3(SL_3(\mathbb{C}))$.

\[
R_3(SL_3(\mathbb{C})) = \mathbb{C}[X,Y,P_{12},P_{23},P_{31},P_{21},P_{32},P_{13}] / < XY - P_{12}P_{23}P_{31} + P_{21}P_{32}P_{13} >
\]
The algebra $R_3(SL_3(\mathbb{C}))$

$V(\omega_1) = \mathbb{C}^3$
$V(\omega_1^*) = \Lambda^2(\mathbb{C}^3)$

Here $P_{ij}$ is the invariant bilinear form in $V(\omega_1) \otimes V(\omega_1^*)$, where $\omega_1$ is in the $i$–th place and $\omega_1^*$ is in the $j$–th place.

$X$ and $Y$ are the determinant invariants in $V(\omega_1) \otimes^3$ and $V(\omega_1^*) \otimes^3$, respectively.
The algebra $V_{0,3}(SL_3(\mathbb{C}))$

Can we use these tensors to get a basis of $V_{0,3}(\lambda, \eta, \mu, L)$?

Each generator $X, Y, P_{12}, P_{23}, P_{31}, P_{21}, P_{32}, P_{13}$ has $\nu_\theta$ value 1
The algebra $V_{0,3}(SL_3(\mathbb{C}))$

Can we use these tensors to get a basis of $V_{0,3}(\lambda, \eta, \mu, L)$?

There is a basis of $[V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^{sl_3(\mathbb{C})}$ by monomials in the generators $X, Y, P_{12}, P_{23}, P_{31}, P_{21}, P_{32}, P_{13}$ each member of which has a distinct $v_\theta$ value.

This basis restricts to a basis of $V_{0,3}(\lambda, \eta, \mu, L)$. 

The algebra $V_{0,3}(SL_3(\mathbb{C}))$

This can be used to construct a presentation of $V_{0,3}(SL_3(\mathbb{C}))$.

$$V_{0,3}(SL_3(\mathbb{C})) = \mathbb{C}[Z, X, Y, P_{12}, P_{23}, P_{31}, P_{21}, P_{32}, P_{13}] / \langle ZXY - P_{12}P_{23}P_{31} + P_{21}P_{32}P_{13} \rangle$$
The algebra $V_{0,3}(SL_3(\mathbb{C}))$
The algebra $V_{0,3}(SL_3(\mathbb{C}))$
Toric degenerations of $V_{0,3}(SL_3(\mathbb{C}))$

\[ ZXY - P_{12}P_{23}P_{31} \]

\[ ZXY + P_{21}P_{32}P_{13} \]

\[ P_{21}P_{32}P_{13} - P_{12}P_{23}P_{31} \]
Polyhedral counting rules: $SL_3(\mathbb{C})$

\[ \nabla_{0,3}(\lambda, \mu, \eta, L) = L - \max\{\lambda_1 + \lambda_2, \mu_1 + \mu_2, \eta_1 + \eta_2, L_1, L_2\} + 1 \]

\[ L_1 = \frac{1}{3}(2(\lambda_1 + \mu_1 + \eta_1) + \lambda_2 + \mu_2 + \eta_2) - \min\{\lambda_1, \mu_1, \eta_1\} \]

\[ L_2 = \frac{1}{3}(2(\lambda_2 + \mu_2 + \eta_2) + \lambda_1 + \mu_1 + \eta_1) - \min\{\lambda_2, \mu_2, \eta_2\}. \]
The multigraded Hilbert scheme for $V_{0,3}$
Toric degenerations of $V_{C,\vec{p}}(SL_3(\mathbb{C}))$

For each trivalent graph $\Gamma$, there are $3^{|V(\Gamma)|}$ toric degenerations of $V_{C,\vec{p}}(SL_3(\mathbb{C}))$. 
For \((\mathbb{P}^1, \vec{p})\) generic, the algebra \(V_{\mathbb{P}^1, \vec{p}}(SL_3(\mathbb{C}))\) is generated by the \(3^{n-1}\) conformal blocks of level 1, and has relations generated by homogenous forms of degree 2, 3.
For \((C, \vec{p})\) generic with \(g = 1\), the algebra \(V_{C, \vec{p}}(SL_3(\mathbb{C}))\) is generated by conformal blocks of level 1, 2, 3.
Possible elements of a generalization

-The tensor product invariants of Howe, Lee; Berenstein, Zelevinsky.

-valuations suggest a role for tropical geometry
Some mathematical biology


-Phylogenetic variety: algebraic variety cut out by binomial equations which vanish on marginal probabilities from a phylogenetic statistical model.

-these are tools for reconstructing ancestral relationships
Some mathematical biology

- Given an oriented, rooted graph $\Gamma$, each vertex $v \in V(\Gamma)$ receives a (possibly $k > 0$ ary) random variable.
- Put probability distribution at the root vertex.
- Each edge $e \in E(\Gamma)$ has a transition matrix $A(e)$, used to recursively compute a distribution at each subsequent vertex.
- The resulting marginal probabilities at the leaves are forced to satisfy equations determined by the matrices $A(e)$. 
Some mathematical biology

One source of matrices $A(v)$ are finite Abelian groups $G$. For each $G$ there is a phylogenetic ”group based” model which determines a binomial ideal $I_{G,T}$ for every structure tree $T$.

For $G = \mathbb{Z}/2\mathbb{Z}$, $I_{T,\mathbb{Z}/2\mathbb{Z}}$ is the ideal defining $\mathbb{C}[P_T(1)]$
Let $A_{\mathcal{T},G}$ denote the corresponding affine semigroup algebra.

[Kubjas, M]: $A_{\mathcal{T},\mathbb{Z}/3\mathbb{Z}}$ is a natural quotient of $\left[\bigotimes_{v \in V(\mathcal{T})} V_{0,3}(SL_3(\mathbb{C}))\right]^{T\mathcal{T}}$

$A_{\mathcal{T},\mathbb{Z}/m\mathbb{Z}}$ is a natural sub-quotient of $\left[\bigotimes_{v \in V(\mathcal{T})} V_{0,3}(SL_m(\mathbb{C}))\right]^{T\mathcal{T}}$
Thankyou!