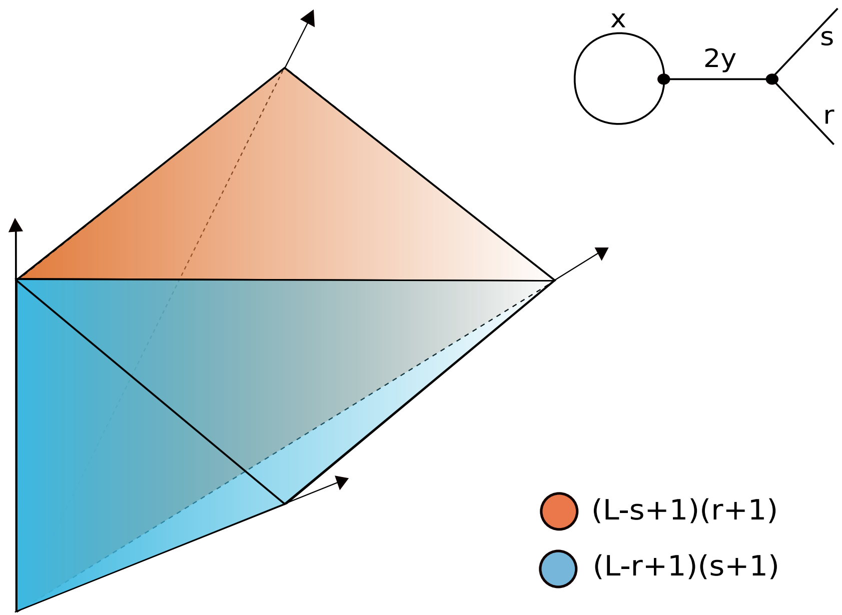
Combinatorial commutative algebra of conformal blocks



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Outline of Talks

Talk 1: Review combinatorics and commutative algebra attached to conformal blocks, introduce flat degenerations of coordinate rings of moduli of principal bundles.

Talk 2: The $SL_2(\mathbb{C})$ case; structure of conformal blocks polytopes; combinatorial commutative algebra of $SL_2(\mathbb{C})$ conformal blocks.

Talk 3: The $SL_3(\mathbb{C})$ case; combinatorics of tensors; relationship between conformal blocks and mathematical biology.

Review from talk I

- $(C, \vec{p}) \in \bar{\mathcal{M}}_{g,n}$: a stable, $n-$marked curve of genus g.

- $\vec{r} = (r_1, \ldots, r_n)$: an $n-$tuple of $sl_2(\mathbb{C})$ dominant weights.

- $V_{C,\vec{p}}(\vec{r}, L)$: the space of $sl_2(\mathbb{C})$ conformal blocks on $(C, \vec{p})$ with weight data $(\vec{r}, L)$

- $\mathbb{V}_{g,n}(\vec{r}, L) = dim[V_{C,\vec{p}}(\vec{r}, L)]$

Review from talk I

- $\mathcal{M}_{C,\vec{p}}(SL_2(\mathbb{C}))$: the moduli of quasi-parabolic $SL_2(\mathbb{C})$ principal bundles on $(C, \vec{p})$.

- $\mathcal{L}(\vec{r}, L)$: The line bundle on $\mathcal{M}_{C,\vec{p}}(SL_2(\mathbb{C}))$ with weight data $(\vec{r}, L)$.

- $V_{C,\vec{p}}(SL_2(\mathbb{C})) = Cox(\mathcal{M}_{C,\vec{p}}(SL_2(\mathbb{C})))$.

$$H^0(\mathcal{M}_{C,\vec{p}}(SL_2(\mathbb{C})), \mathcal{L}(\vec{r}, L)) = V_{C,\vec{p}}(\vec{r}, L)$$

Review from talk I

$$V_{C,\vec{p}}(SL_2(\mathbb{C})) = \bigoplus_{\vec{r},L} V_{C,\vec{p}}(\vec{r}, L)$$

$$R_{C,\vec{p}}(\vec{r}, L) = \bigoplus_{N \geq 0} V_{C,\vec{p}}(N\vec{r}, NL)$$

Tangential remark

Let $L >$ critical level, then

$$Proj(R_{\mathbb{P}^1,\vec{p}}(\vec{r}, L)) = [\mathbb{P}^1]^n //_{\vec{r}} SL_2(\mathbb{C})$$

Review from talk I

Questions:

-What generates the conformal blocks?

-What relations hold among these generators?

-How do we count the conformal blocks?

Review from talk I

Questions:

- What generates $V_{C,\vec{p}}(SL_2(\mathbb{C}))$?

-What relations hold among these generators?

-What is the multigraded Hilbert function of $V_{C,\vec{p}}(SL_2(\mathbb{C}))$?

Review from talk I

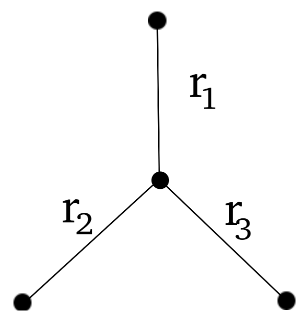
For every trivalent graph with first Betti number g and n leaves, there is a flat degeneration

$$V_{C,\vec{p}}(SL_2(\mathbb{C})) \Rightarrow [\bigotimes_{v \in V(\Gamma)} V_{0,3}(SL_2(\mathbb{C}))]^{T_\Gamma}$$

-Reduces to the $g = 0, n = 3$ case.

Review from talk I

Proposition [Quantum Clebsch-Gordon rule] :
The dimension of $V_{0,3}(r_1, r_2, r_3, L)$ is either 1 or 0. It is dimension 1 if and only if $r_1 + r_2 + r_3$ is even, $\leq 2L$, and r_1, r_2, r_3 are the side-lengths of a triangle



Review from talk I

[Tsuchiya, Ueno, Yamada]: The space $V_{0,3}(\lambda, \eta, \mu, L)$ can be identified with a subspace of $[V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^{\mathfrak{g}}$

$\theta(sl_2(\mathbb{C})) \subset \mathfrak{g}$

$W_L(\lambda, \eta, \mu) = \bigoplus_{i+j+k \leq 2L} W_{\lambda,i} \otimes W_{\eta,j} \otimes W_{\mu,k}$.

$$V_{0,3}(\lambda, \eta, \mu, L) = W_L(\lambda, \eta, \mu) \cap [V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^{\mathfrak{g}}$$

Review from talk I

[classical Clebsch-Gordon rule]: The space $[V(r_1) \otimes V(r_2) \otimes V(r_3)]^{sl_2(\mathbb{C})}$ is dimesion 0, or 1. It is dimension 1 if and only if $r_1 + r_2 + r_3$ is even, and r_1, r_2, r_3 are the side-lengths of a triangle

-$V(r) = Sym^r(\mathbb{C}^2)$

-follows from computation on weight diagrams.

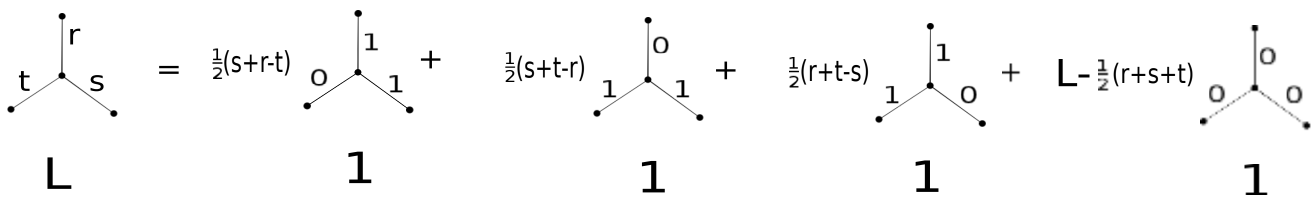
Structure of $0, 3$ algebra

$$V_{0,3}(SL_2(\mathbb{C})) = \bigoplus_{r_1, r_2, r_3, L \geq 0} V_{0,3}(r_1, r_2, r_3, L)$$

-Generated by

$V_{0,3}(1, 1, 0, 1)$, $V_{0,3}(1, 0, 1, 1)$, $V_{0,3}(0, 1, 1, 1)$, $V_{0,3}(0, 0, 0, 1)$

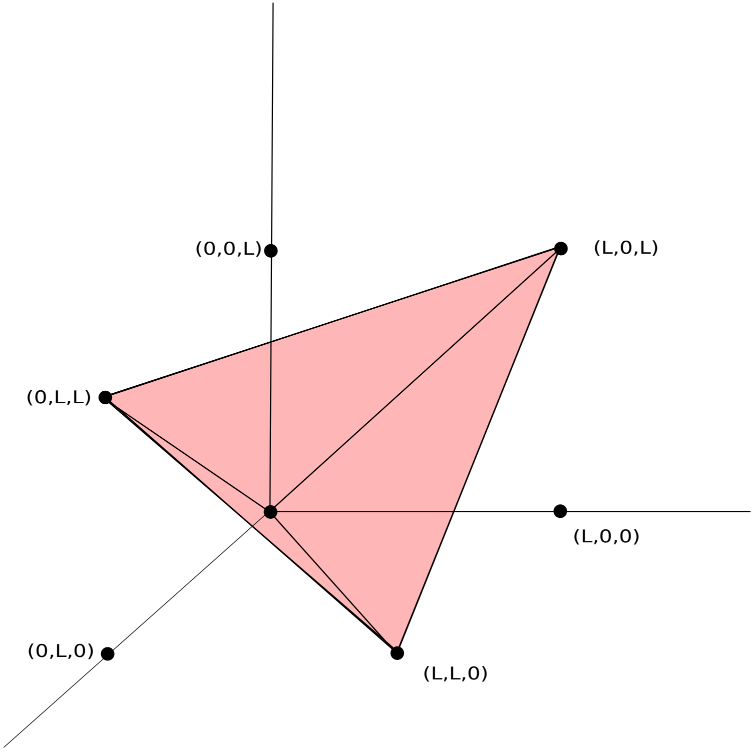
Structure of $0, 3$ algebra



$$P_3(1) = conv\{(1011), (1101), (0111), (0001)\}$$

Structure of $0,3$ algebra

$$V_{0,3}(SL_2(\mathbb{C})) = \mathbb{C}[P_3(1)]$$



$\frac{1}{2}$

Structure of g, n algebra

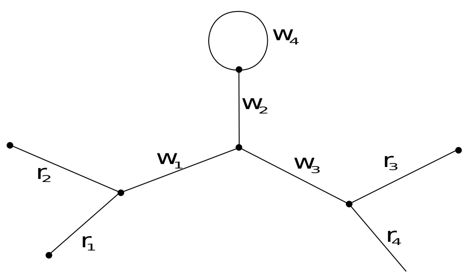
-Use the $0, 3$ case to build the general case via the degeneration theorem.

- Must describe the algebra

$$[\bigotimes_{v \in V(\Gamma)} V_{0,3}(SL_2(\mathbb{C}))]^{T_\Gamma} \subset \bigotimes_{v \in V(\Gamma)} V_{0,3}(SL_2(\mathbb{C})).$$

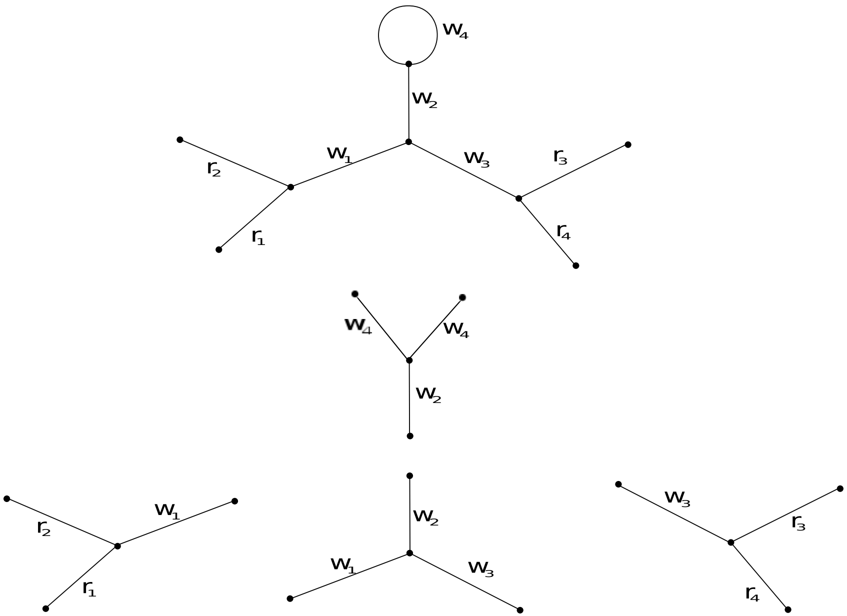
Structure of g, n algebra

Choose your favorite trivalent graph with first Betti number g and n leaves.



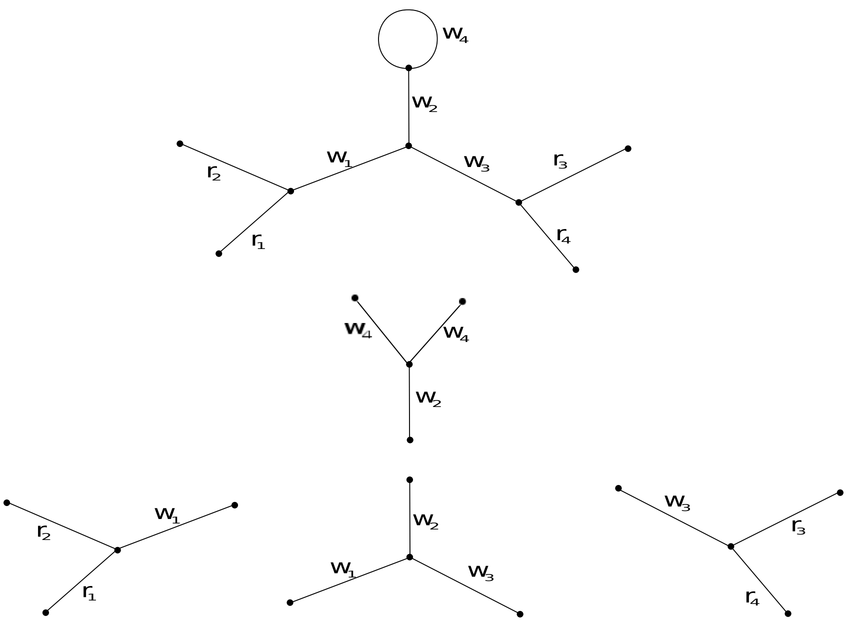
Structure of g, n algebra

Cut all non-leaf edges in half.



Structure of g, n algebra

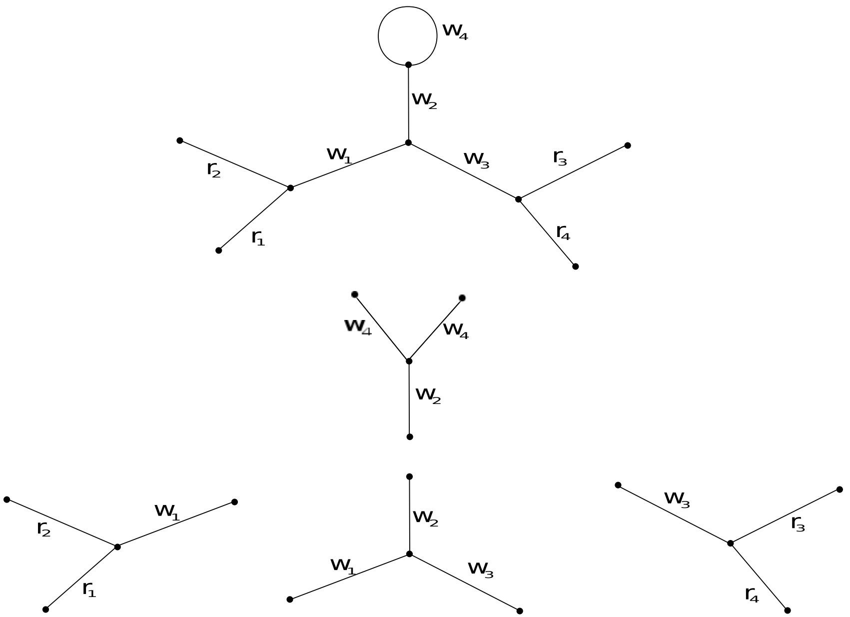
Cut all non-leaf edges in half.



Each of the resulting trinodes is assigned a copy of $V_{0,3}(SL_2(\mathbb{C}))$.

Structure of g, n algebra

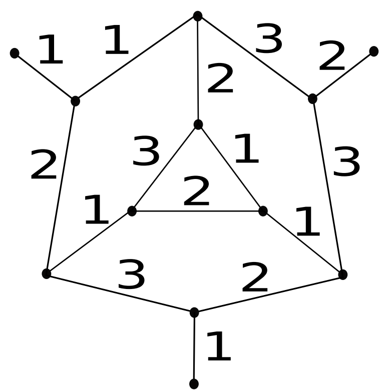
This is the tensor product $\bigotimes_{v \in V(\Gamma)} V_{0,3}(SL_2(\mathbb{C}))$.



$[\bigotimes_{v \in V(\Gamma)} V_{0,3}(SL_2(\mathbb{C}))]^{T_\Gamma}$ is then the subalgebra where the levels at each trinode match, and the shared edges match.

The polytope $P_\Gamma(L)$

Definition:For Γ a trivalent graph of genus g with n marked points we define $P_\Gamma(L)$ to be the polytope given by non-negative integer weightings of the edges of Γ which satisfy the Quantum Clebsch-Gordon rules at each trinode with respect to level L.



Structure of g, n algebra

There is a flat degeneration $V_{C,\vec{p}}(SL_2(\mathbb{C})) \Rightarrow \mathbb{C}[P_\Gamma(1)]$

The polytope $P_\Gamma(\vec{r}, L)$

Definition:For Γ a trivalent graph of genus g with n marked points we define $P_\Gamma(\vec{r}, L)$ to be the polytope given by non-negative integer weightings of the edges of Γ which satisfy the Quantum Clebsch-Gordon rules at each trinode with respect to level L, and weight the $i-$th leaf of Γ with r_i.

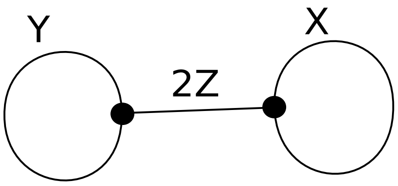
Let $\pi : P_\Gamma(L) \to \mathbb{R}^n$ be the map which forgets everything but the weights on the leaves. Then $P_\Gamma(\vec{r}, L) = \pi^{-1}(\vec{r})$.

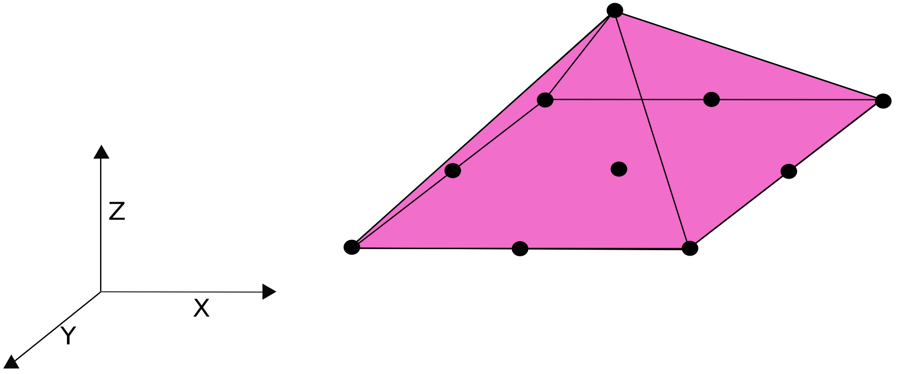
Structure of projective coordinate ring $R_{C,\vec{p}}(\vec{r}, L)$

There is a flat degeneration $R_{C,\vec{p}}(\vec{r}, L) \Rightarrow \mathbb{C}[P_\Gamma(\vec{r}, L)]$

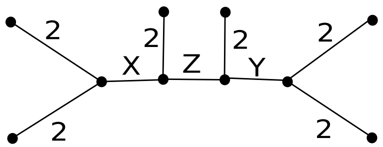
$\mathbb{V}_{g,n}(\vec{r}, L)$ is equal to the number of lattice points in the polytope $P_\Gamma(\vec{r}, L)$.

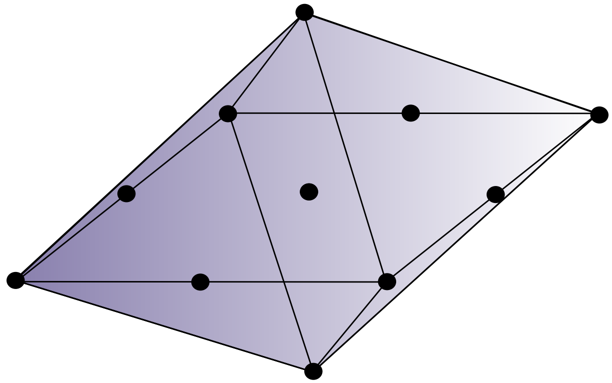
Examples: $P_{2,0}(2)$



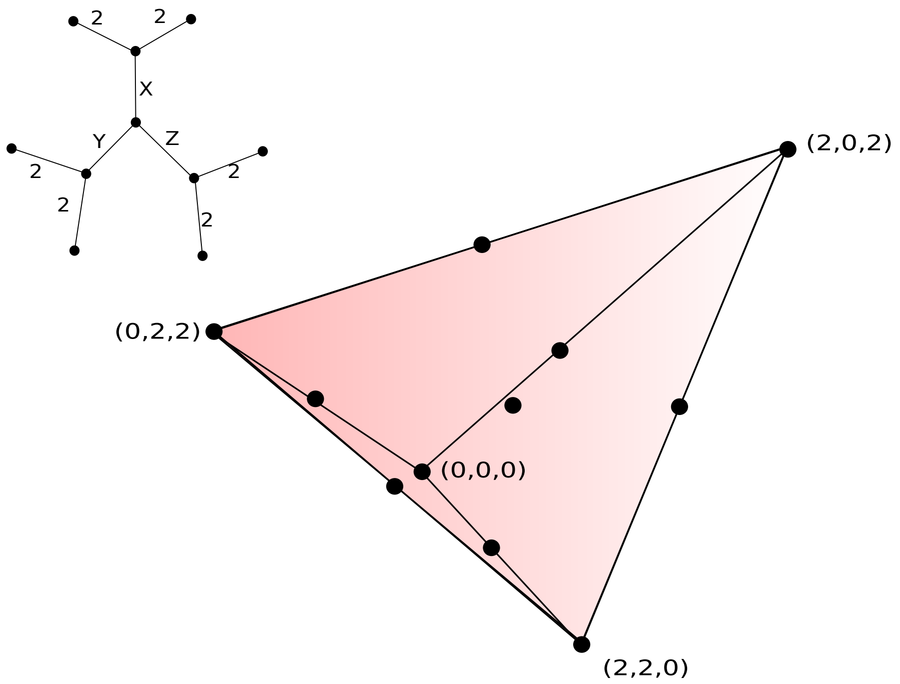


Examples: $P_{0,6}(2,2,2,2,2,2,4)$

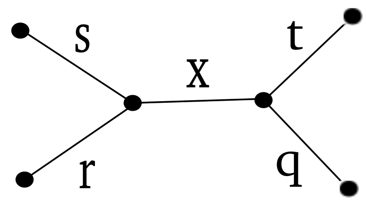


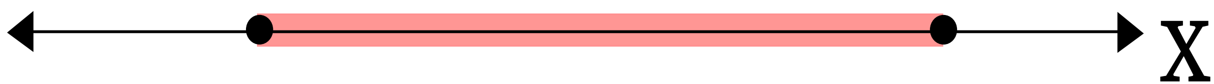


Examples: $P_{0,6}(2, 2, 2, 2, 2, 2, 4)$

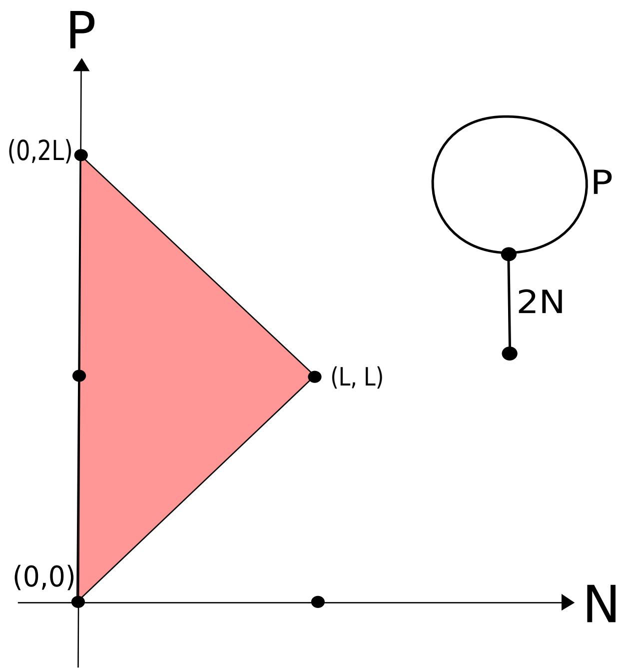


Examples: $P_{0,4}(s, r, t, q, L)$

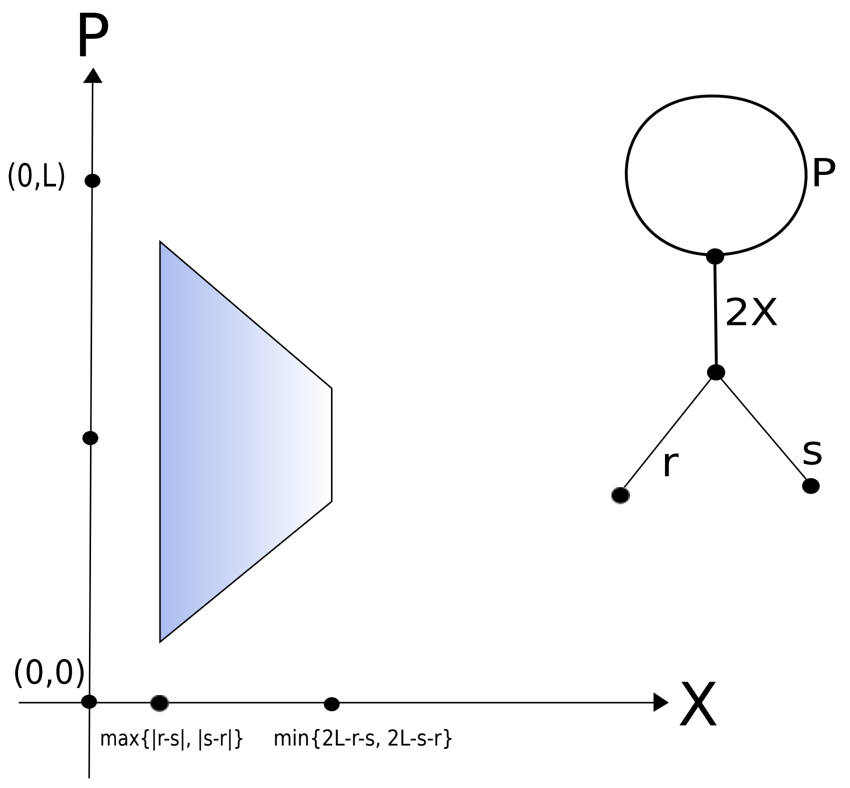




Examples: $P_{1,1}(2L)$



Examples: $P_{1,2}(r, s, L)$



Affine semigroup algebras

-always finitely generated (Gordon's Lemma),

-always cut out by binomial relations,

-always Cohen-Macaulay,

-software: 4ti2, polymake, Macaulay 2.

degeneration to an affine semigroup algebra

-Can lift generators and relations.

-Can lift the multigraded Hilbert function.

-Implies Cohen-Macaulay

Commutative algebra of $V_{C,\vec{p}}(SL_2(\mathbb{C}))$

[Buczynska, Buczynski, Kubjas, Michalek]: For any graph Γ the semigroup algebra $\mathbb{C}[P_\Gamma(1)]$ is generated by elements of degree $\leq g+1$.

For generic $(C, \vec{p})$, the algebra $V_{C,\vec{p}}(SL_2(\mathbb{C}))$ is generated by conformal blocks of level $\leq g+1$.

Commutative algebra of $V_{C,\vec{p}}(SL_2(\mathbb{C}))$

[M]: For a special graph $\Gamma(g,n)$, the semigroup algebra $\mathbb{C}[P_{g,n}(1)]$ is generated by elements of degree ≤ 2, and has relations generated by forms of degree ≤ 4.

[M]: For generic $(C,\vec{p})$ the algebra $V_{C,\vec{p}}(SL_2(\mathbb{C}))$ is generated by conformal blocks of level ≤ 2, and the relations on these generators are generated by forms of degree ≤ 4.

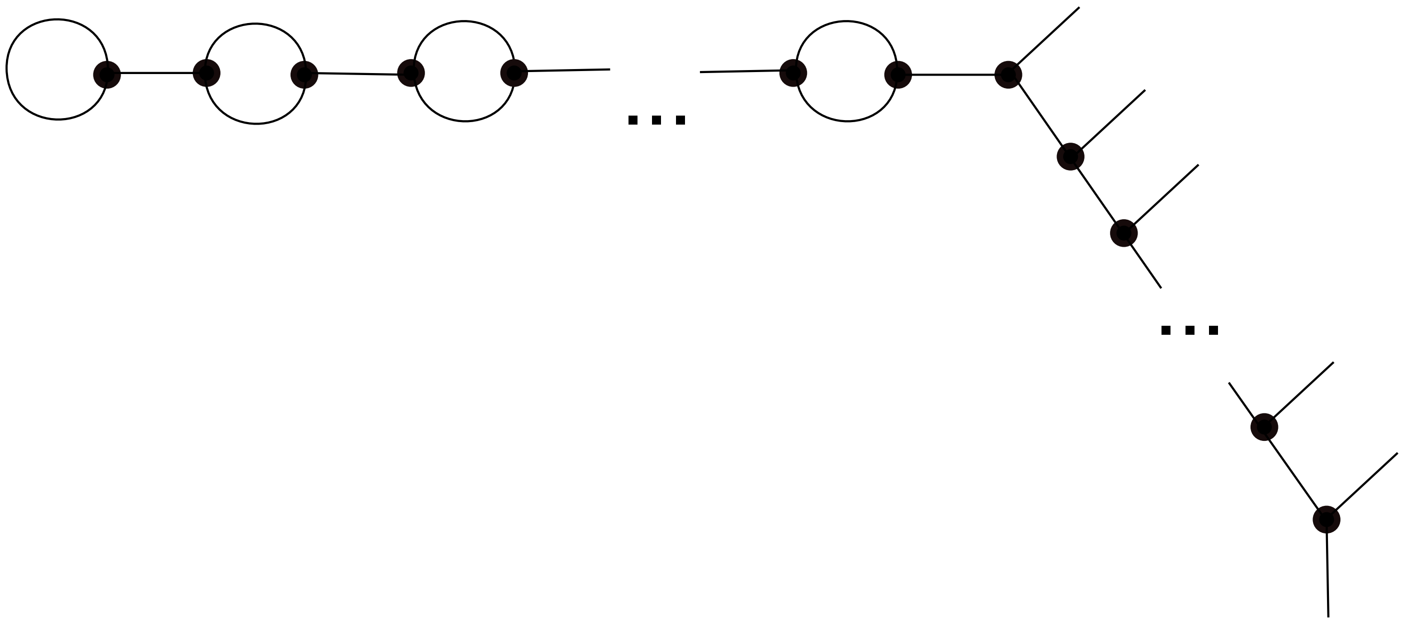
Commutative algebra of $V_{C,\vec{p}}(SL_2(\mathbb{C}))$

[Abe]: Showed generation in degree ≤ 2 in the $n = 0$ case.

[Castravet, Tevelev]: Showed generation in degree 1 in the $g = 0$ case.

[Buczynska, Wiesniewski]: Showed generation in degree 1 and relations in degree 2 for $\mathbb{C}[P_{\mathcal{T}}(1)]$, where $\mathcal{T}$ is a tree.

Commutative algebra of $V_{C,\vec{p}}(SL_2(\mathbb{C}))$



The graph $\Gamma(g, n)$.

Commutative algebra of $R_{C,\vec{p}}(\vec{r}, L)$

Passing to a sub-algebra often makes things more difficult.

[M]: For special graphs Γ, the semigroup algebra $\mathbb{C}[P_\Gamma(2\vec{r}, 2L)]$ is generated by elements of degree 1, and the associated binomial ideal has a quadratic, square-free Gröbner basis.

-This says that $P_\Gamma(2\vec{r}, 2L)$ is a $normal$ lattice polytope.

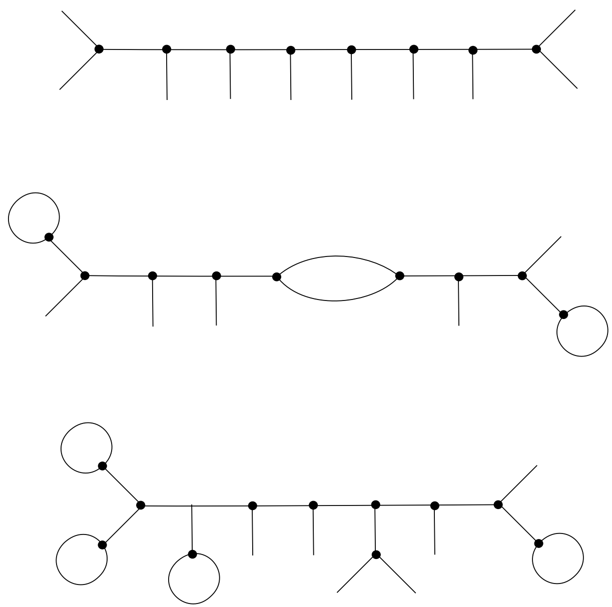
For generic $(C, \vec{p})$, the algebra $R_{C,\vec{p}}(2\vec{r}, 2L)$ is generated in degree 1 and is Koszul.

Commutative algebra of $R_{C,\vec{p}}(\vec{r}, L)$

The square $\mathcal{L}(\vec{r}, L)^{\otimes 2}$ of an effective line bundle on the moduli $\mathcal{M}_{C,\vec{p}}(SL_2(\mathbb{C}))$ has a Koszul projective coordinate ring.

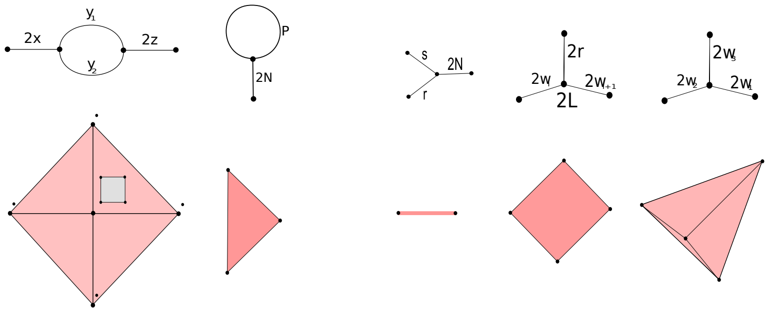
Commutative algebra of $R_{C,\vec{p}}(\vec{r}, L)$

Special graphs for the theorem on $R_{C,\vec{p}}(2\vec{r}, 2L)$:



Commutative algebra of $R_{C,\vec{p}}(\vec{r}, L)$

The polytopes for the special graphs are made by fiber products of the following building blocks over $[0, L]$.



Combinatorics of polytopes

$\pi_1 : P \to D$

$\pi_2 : Q \to D$

$$P \times_D Q = \{(v, w) | \pi_1(v) = \pi_2(w)\}$$

This polytope behaves well when P, Q, D behave well.

Combinatorics of polytopes

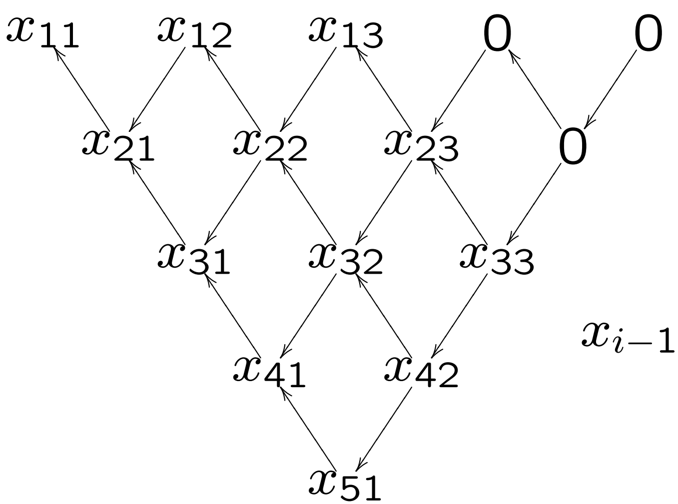
Each building block polytope is "balanced."

Definition: A polytope P is balanced if for each lattice point $w \in L \circ P \cap \mathbb{L}$ there is a combination of ceilings and floors $\widehat{w}$ on the entries of $\frac{1}{L}w$ such that $\widehat{w} \in P \cap \mathbb{L}$ and $w - \widehat{w} \in (L-1) \circ P \cap \mathbb{L}$.

-This is equivalent to requiring that intersection $(C + v) \cap P$ be a normal polytope for every lattice translate $(C + v)$ of the fundamental domain C of the lattice $\mathbb{L}$.

Combinatorics of polytopes

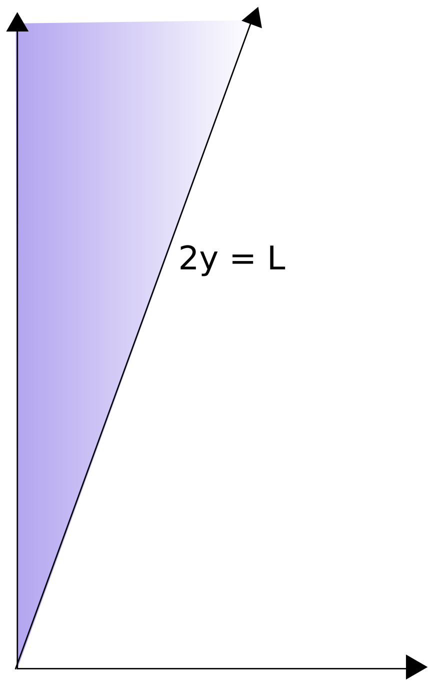
Another balanced polytope: $GT(m,n,L)$

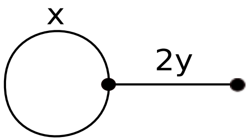


$$x_{i-1,1} - x_{i,n-i} \leq L$$

$$\bigoplus_{\vec{r},L} V_{\mathbb{P}^1,\vec{p}}(\vec{r},L) = Cox(\mathcal{M}_{\mathbb{P}^1,\vec{p}}(SL_m(\mathbb{C}),\vec{P}))$$

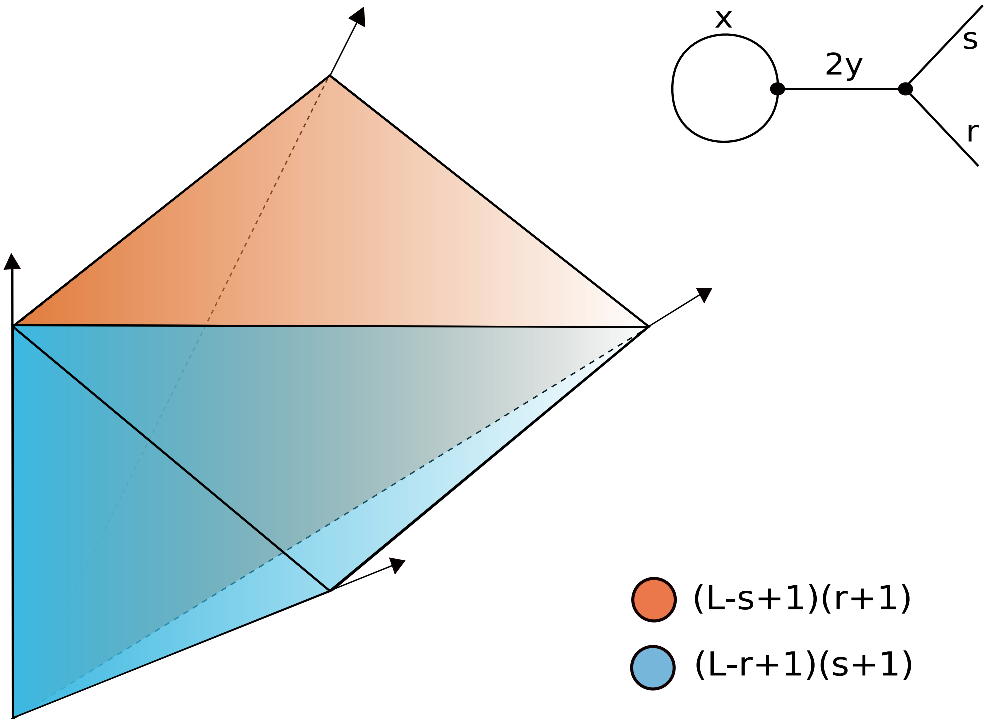
Ex: Multigraded Hilbert function of $V_{1,1}(SL_2(\mathbb{C}))$





L - 2y +1

Ex: Multigraded Hilbert Function of $V_{1,2}(SL_2(\mathbb{C}))$



Thankyou!