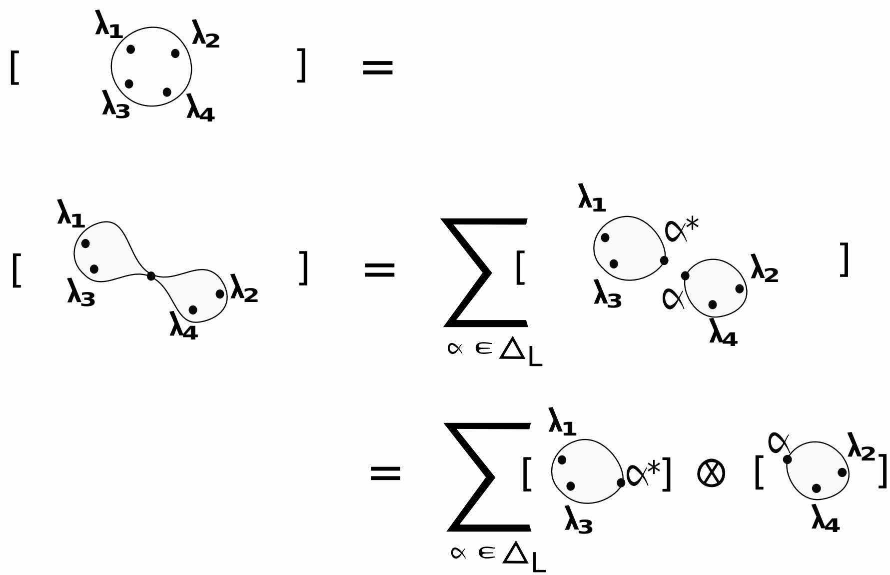
Combinatorial commutative algebra of conformal blocks



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Outline of Talks

Talk 1: Review combinatorics and commutative algebra attached to conformal blocks, introduce flat degenerations of coordinate rings of moduli of principal bundles.

Talk 2: The $SL_2(\mathbb{C})$ case; structure of conformal blocks polytopes; combinatorial commutative algebra of $SL_2(\mathbb{C})$ conformal blocks.

Talk 3: The $SL_3(\mathbb{C})$ case; combinatorics of tensors; relationship between conformal blocks and mathematical biology.

Notation (everything over $\mathbb{C}$)

$\mathfrak{g}$: Simple Lie algebra.

G: Simple, simply-connected algebraic group ($Lie(G) = \mathfrak{g}$).

λ: dominant weight of $\mathfrak{g}, G$.

Δ: a Weyl chamber of $\mathfrak{g}, G$.

$(C, \vec{p}) \in \bar{\mathcal{M}}_{g,n}$: stable n-marked genus g curve.

Notation

Δ_L: The $L-$restricted part of Δ.

$V_{C,\vec{p}}(\vec{\lambda}, L)$: conformal blocks with weight data $\vec{\lambda}$, $\lambda_i \in \Delta$, and level L.

$\mathcal{V}_{g,n}(\vec{\lambda}, L)$: the dimension of $V_{C,\vec{p}}(\vec{\lambda}, L)$.

$\mathcal{M}_{C,\vec{p}}(G)$: moduli of quasi-parabolic principal G-bundles. The parabolic structure at each marked point will come from a Borel subgroup $B \subset G$ unless indicated otherwise.

$\mathcal{L}(\vec{\lambda}, L)$: line bundle on $\mathcal{M}_{C,\vec{p}}(G)$ with weight data $\vec{\lambda}$, level L.

Conformal Blocks and Principal Bundles

Let $\mathcal{X}(B)$ be the character group of the Borel $B \subset G$.

[Beauville, Laszlo, Sorger]

$$Pic(\mathcal{M}_{C,\vec{p}}(G)) = \mathcal{X}(B)^n \times \mathbb{Z}$$

Conformal Blocks and Principal Bundles

[Kumar, Narasimhan, Ramanathan; Faltings; Beauville, Laszlo, Sorger; Pauly] There is a natural isomorphism

$$H^0(\mathcal{M}_{C,\vec{p}}(G), \mathcal{L}(\vec{\lambda}, L)) = V_{C,\vec{p}}(\vec{\lambda}, L).$$

Definition:

$$R_{C,\vec{p}}(\vec{\lambda}, L) = \bigoplus_{N \geq 0} V_{C,\vec{p}}(N\vec{\lambda}, NL)$$

Conformal Blocks and Principal Bundles

Consequence:

The vector space $V_{C,\vec{p}}(G) = \bigoplus_{\vec{\lambda},L} V_{C,\vec{p}}(\vec{\lambda}, L)$ can be given the structure of a multigraded commutative ring, isomorphic to $Cox(\mathcal{M}_{C,\vec{p}}(G))$.

Innocent questions

-How do we generate all conformal blocks?

-What relations must hold between them?

-How do we count the dimensions of spaces of conformal blocks?

Innocent questions [Commutative algebra]

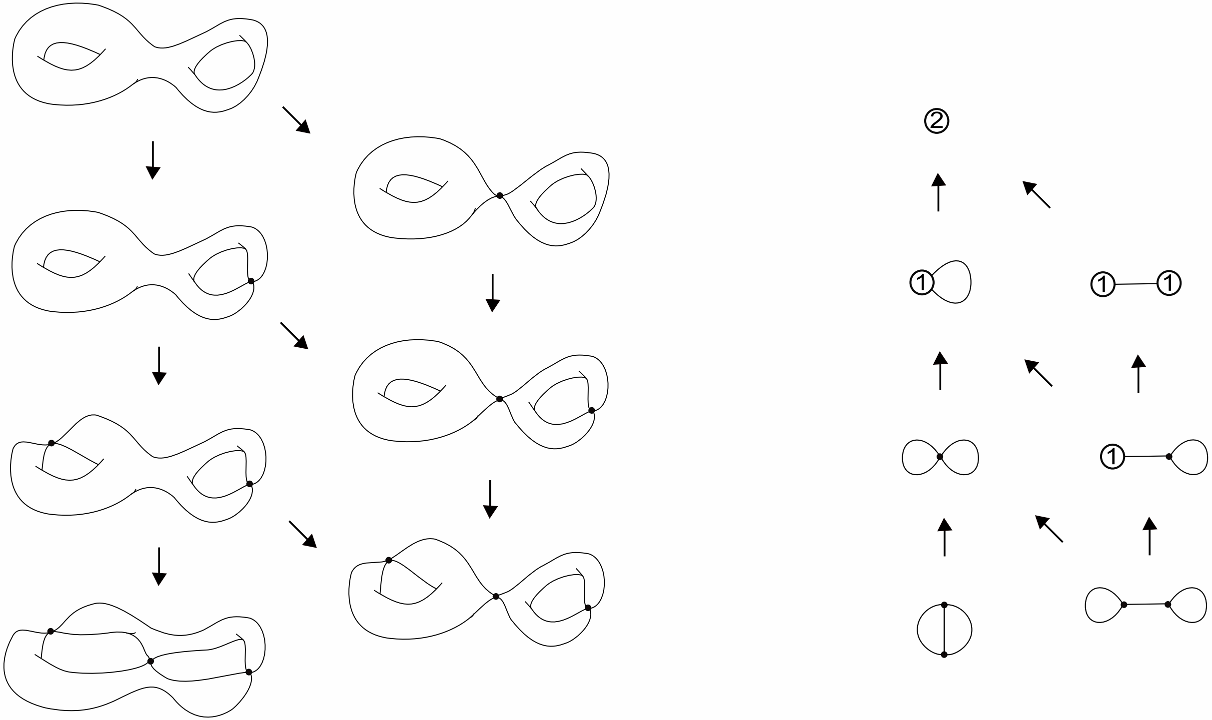
-What multigraded components generate $V_{C,\vec{p}}(G)$?

-What relations hold among these generators?

-What is the multigraded Hilbert function of $V_{C,\vec{p}}(G)$?

The moduli stack $\bar{\mathcal{M}}_{g,n}$

Example: $\bar{\mathcal{M}}_{2,0}$



Let $(C_\Gamma, \vec{p}) \in \bar{\mathcal{M}}_{g,n}$ be the curve associated to the trivalent graph Γ.

Geometry for counting rules

The dimension of $V_{C,\vec{p}}(\vec{\lambda}, L)$ depends only on the data $(\vec{\lambda}, L)$, the genus g, and the number of points n.

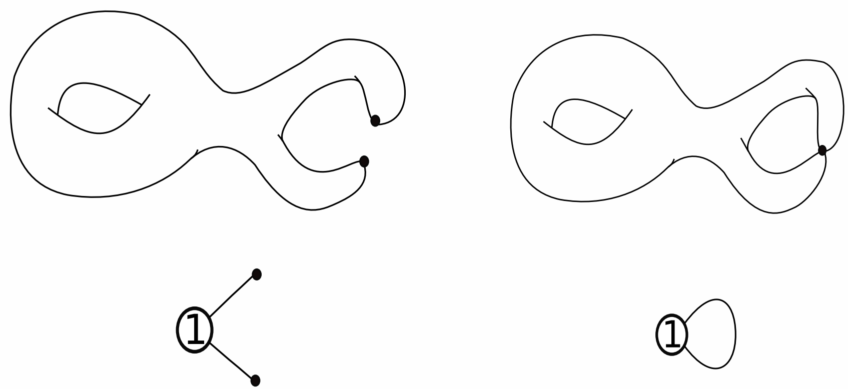
Theorem [Tsuchiya, Ueno, Yamada]:

For every $\vec{\lambda}, L$ there is a vector bundle $V(\vec{\lambda}, L)$ on $\bar{\mathcal{M}}_{g,n}$, with fiber over $(C, \vec{p})$ equal to $V_{C,\vec{p}}(\vec{\lambda}, L)$.

-Note that $(C, \vec{p})$ need not be smooth.

Geometry for counting rules

For $(C, \vec{p})$ a stable curve with a doubled point $q \in C$, there is a normalized stable marked curve $(C', q_1, q_2, \vec{p})$.

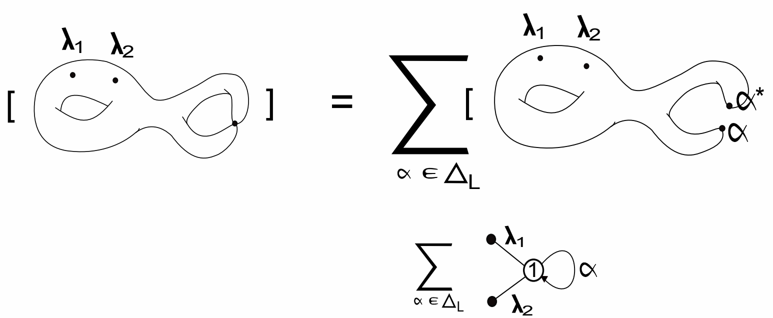


Conformal Blocks: Factorization

For a marked curve $(C,\vec{p})$ with a doubled point q,

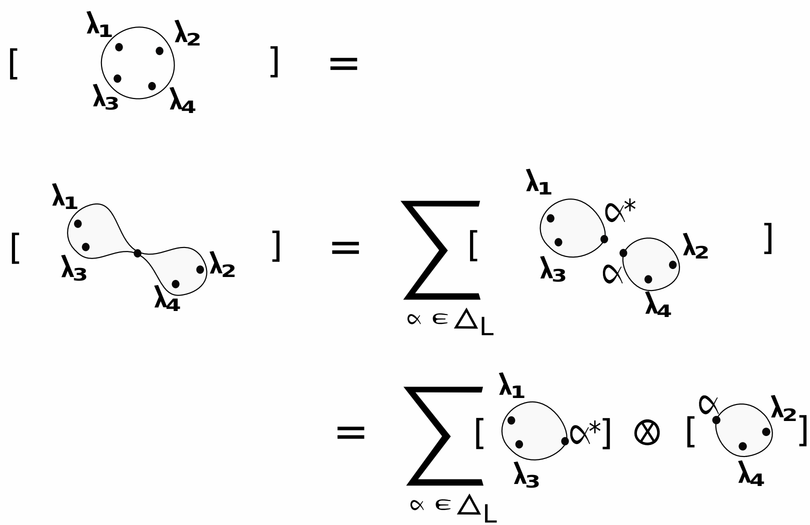
[Tsuchiya, Ueno, Yamada; Faltings]

$$V_{C,\vec{p}}(\vec{\lambda},L) \cong \bigoplus_{\alpha\in\Delta_L} V_{C',q_1,q_2,\vec{p}}(\alpha,\alpha^*,\vec{\lambda},L)$$

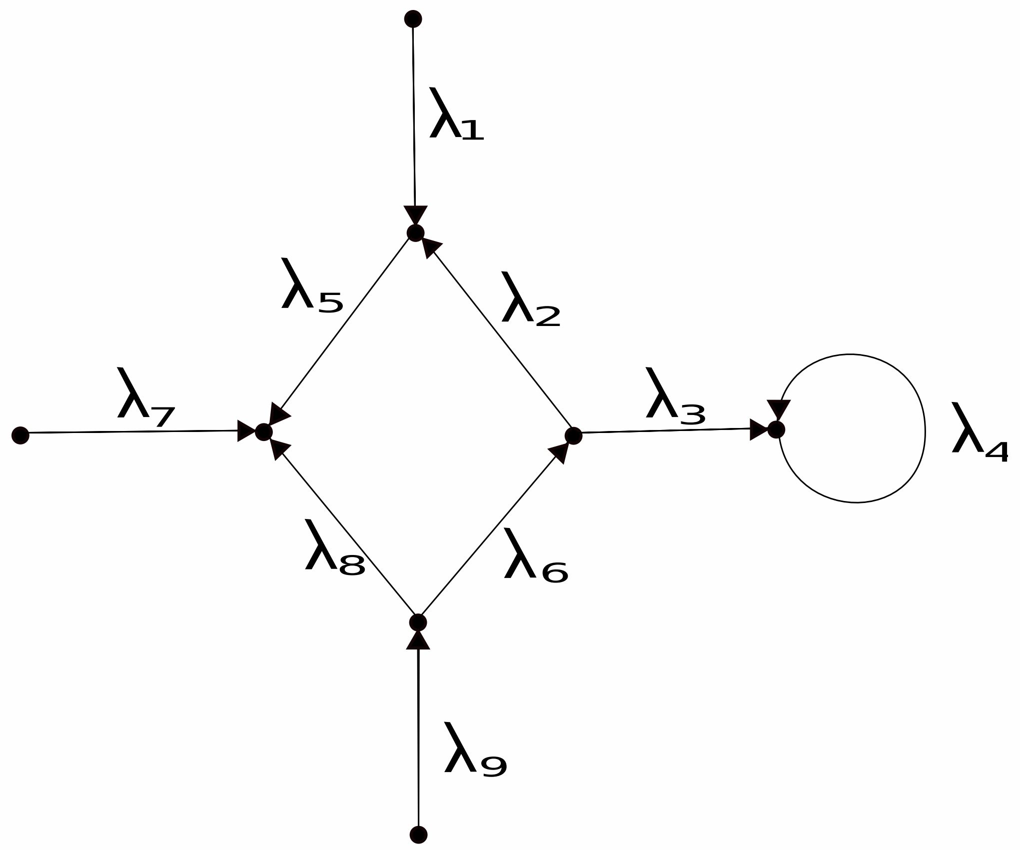


Conformal Blocks: Factorization

The flatness of $V(\vec{\lambda}, L)$ and the factorization rules allow us to pass from general curves to 3−marked genus 0 curves.



Conformal Blocks: Factorization



Combinatorics of conformal blocks

Some terminology:

A rule for counting things: A formula or other method for evaluating the number of things.

A combinatorial rule for counting things: An explicitely integer formula (may involve some minus signs).

A positive counting rule for counting things: An explicitely non-negative integer formula.

A polyhedral counting rule for counting things: Bijection of things with the lattice points in a polytope.

An (by no means complete) overview of counting rules for conformal blocks

A counting rule (The Verlinde Formula):

[Verlinde, Faltings, Beauville]:

$$\mathbb{V}_{g,n}(\vec{\lambda}, L) = |T_L|^{g-1} \sum_{\mu \in \Delta_L} Tr_{\vec{\lambda}}(exp(2\pi i \frac{(\mu+\rho)}{L+h^\vee})) \prod_{\alpha} |2sin(\pi \frac{(\alpha|\mu+\rho)}{L+h^\vee})|^{2-2g}$$

-Reduce to the $g = 0, n = 3$ point case, use character theory of the Fusion algebra.

$$X_{\lambda,L} \boxtimes X_{\eta,L} = \sum_{\alpha \in \Delta_L} \mathbb{V}_{0,3}(\lambda, \eta, \alpha^*, L) X_{\alpha,L}$$

A combinatorial rule (The Kac-Walton Formula)

$\widehat{W}-$ the affine Weyl group.

$$\mathbb{V}_{0,3}(\lambda,\eta,\mu,L) = \sum_{w\in\widehat{W}, w\circ\tau=\mu} \epsilon(w) c_{\lambda,\eta,\tau}$$

$c_{\lambda,\eta,\tau} = dim([V(\lambda)\otimes V(\eta)\otimes V(\tau)]^{\mathfrak{g}})$

$w\circ\tau = w(\tau+\rho)-\rho$

$\rho = \sum \omega_i$

A combinatorial rule

Korff and Stroppel have found a combinatorial "quantum Littlewood-Richardson rule" by presenting the fusion algebra in type A using an affine version of the Plactic algebra. The resulting rule is distinct from the Kac-Walton formula.

Morse and Schilling have found combinatorial rules for conformal blocks with special kinds of weight data, expressed as "cylindrical tableaux." This involves Kostka matrices, and gives a positive counting rule.

A positive counting rule

Buch, Kresch, Tamvakis have formulated a positive, conjectural counting rule by linking the dimension of the space of conformal blocks in type A to a classical intersection number on a two-step flag variety.

This leads to a formulation in terms of A. Knutson's puzzles.

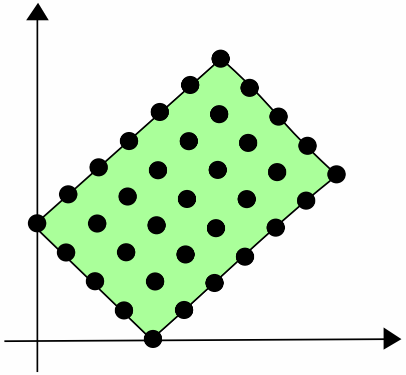
Polyhedral counting rules: definitions

$\mathbb{L} \subset \mathbb{R}^n$ a lattice, $A \in M_{m,n}(\mathbb{R})$, a rational matrix, $b \in \mathbb{R}^m$

$$P_{A,b} = \{v | Av \leq b\}$$

Polytope: a compact solution set to a set of inequalities.

$P_{A,b} \cap \mathbb{L}$ - the lattice points in the polytope.

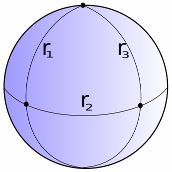


Polyhedral counting rules: $SL_2(\mathbb{C})$

In the case $G = SL_2(\mathbb{C})$, the spaces $V_{0,3}(r_1, r_2, r_3, L)$ are dimension 1 or 0.

Proposition [Quantum Clebsch-Gordon rule] :

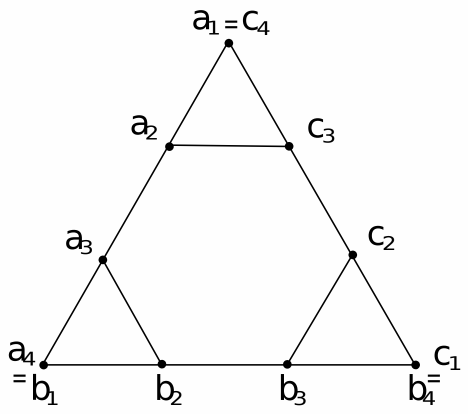
The dimension of $V_{0,3}(r_1, r_2, r_3, L)$ is either 1 or 0. It is dimension 1 if and only if $r_1 + r_2 + r_3$ is even, $\leq 2L$, and r_1, r_2, r_3 are the side-lengths of a triangle



Polyhedral counting rules: $SL_3(\mathbb{C})$

$\mathbb{V}_{0,3}(\lambda, \eta, \mu, L) =$ The number of diagrams below with

- all ≥ 0 integer entries,
- $(a_1 + a_2, a_3 + a_4) = \lambda$, $(b_1 + b_2, b_3 + b_4) = \eta$, $(c_1 + c_2, c_3 + c_4) = \mu$,
- $a_2 + a_3 = c_2 + b_3$, $b_2 + b_3 = a_2 + c_3$, $c_2 + c_3 = b_2 + a_3$,
- $a_1 + b_1 + c_1 + ... \leq L$



Polyhedral counting rules: $SL_3(\mathbb{C})$

$$\mathbb{V}_{0,3}(\lambda, \mu, \eta, L) = L - max\{\lambda_1 + \lambda_2, \mu_1 + \mu_2, \eta_1 + \eta_2, L_1, L_2\} + 1$$

$$L_1 = \frac{1}{3}(2(\lambda_1 + \mu_1 + \eta_1) + \lambda_2 + \mu_2 + \eta_2) - min\{\lambda_1, \mu_1, \eta_1\}$$

$$L_2 = \frac{1}{3}(2(\lambda_2 + \mu_2 + \eta_2) + \lambda_1 + \mu_1 + \eta_1) - min\{\lambda_2, \mu_2, \eta_2\}.$$

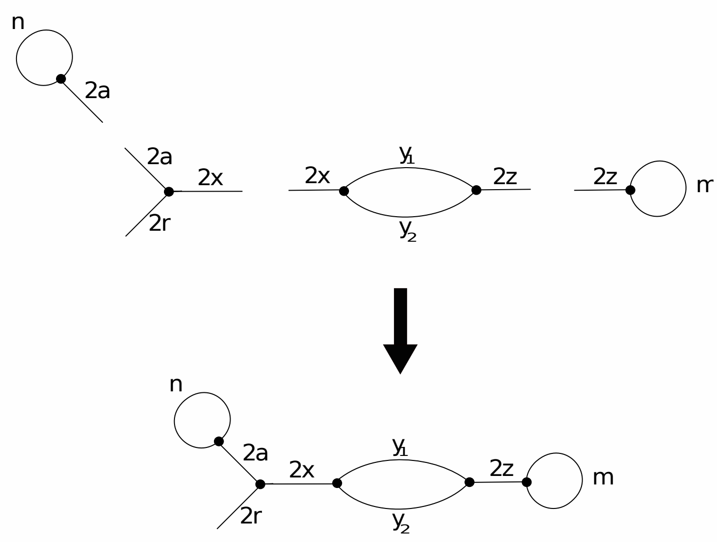
-Senechal, Mathieu, Kirillov, and Walton; [M]

Polyhedral counting rules

A polyhedral counting rule in the $g = 0, n = 3$ case yields a polyhedral counting rule in all cases via toric fiber products.

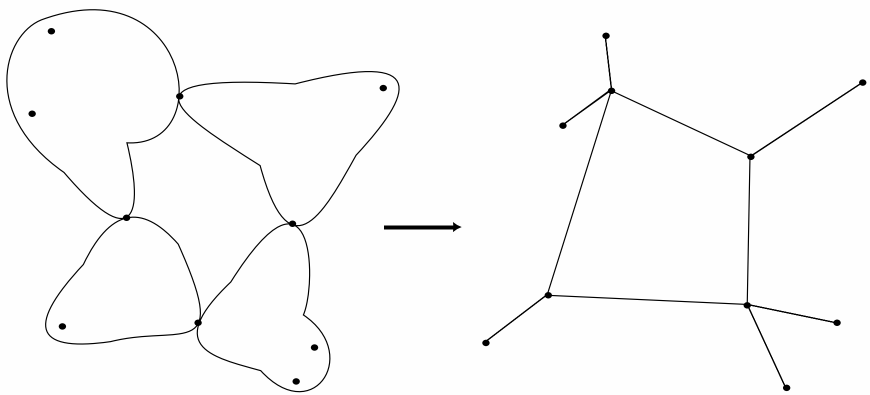
$\pi_1 : P \to D$, $\pi_2 : Q \to D$,

$P \times_D Q = \{(v, w) | v \in P, w \in Q, \pi_1(v) = \pi_2(w)\}$



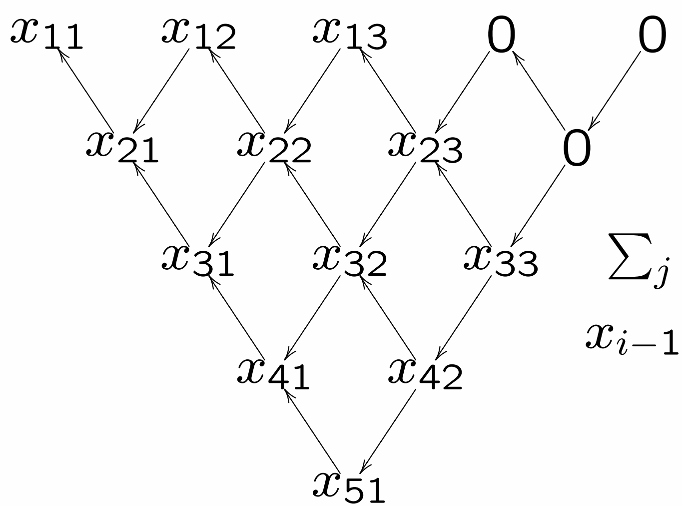
Polyhedral counting rules

Given a polyhedral rule to evaluate $\mathbb{V}_{0,3}$, there is a distinct polyhedral counting rule for evaluating $\mathbb{V}_{g,n}$ for each special curve $(C_\Gamma, \vec{p}) \in \bar{\mathcal{M}}_{g,n}$



Polyhedral Counting rules: $k-$Pieri rule

For $G = SL_m(\mathbb{C})$, $\lambda_i = r_i\omega_1$, the dimension of $V_{\mathbb{P}^1,\vec{p}}(\vec{r}\omega_1, L)$ is the number of lattice points in $GT(\vec{r}, L)$, given by non-negative integer fillings of the following interlacing diagram.



$$\sum_j x_{ij} = \sum_{k\leq i} r_k$$

$$x_{i-1,1} - x_{i,n-i} \leq L$$

This is an application of the "$k-$Pieri rule."

Combinatorial commutative algebra

Each polytope $P \subset (\mathbb{R}^n, \mathbb{L})$ has an associated affine semigroup algebra $\mathbb{C}[P]$.

This is constructed by considering the lattice points in the Minkowski sums of P.

$$k \circ P = \{v_1 + \ldots + v_k | v_i \in P\}$$

$$\mathbb{C}[P]_k = \{v | v \in (k \circ P) \cap \mathbb{L}\}$$

$$\mathbb{C}[P]_k \times \mathbb{C}[P]_\ell \to \mathbb{C}[P]_{k+\ell}$$

Combinatorial commutative algebra

These algebras are "nice" (or at least nicer than your average commutative algebra):

-always finitely generated (Gordon's Lemma),

-always cut out by binomial relations,

-always Cohen-Macaulay,

-software: 4ti2, polymake, Macaulay 2.

Combinatorial commutative algebra of conformal blocks

Each of the previous polyhedral counting rules comes with a natural algebraic structure coming from affine semigroup multiplication.

Question: What does this structure have to do with $V_{C,\vec{p}}(G)$?

Combinatorial commutative algebra of conformal blocks

Each of the previous polyhedral counting rules comes with a natural algebraic structure coming from affine semigroup multiplication.

Question: What does this structure have to do with $V_{C,\vec{p}}(G)$?

Strategy: Find a toric degeneration of $V_{C,\vec{p}}(G)$.

-give polyhedral counting rules

-use to study algebraic structure.

Results of Sturmfels-Xu

[Sturmfels, Xu]: For each $\mathcal{T}$ a trivalent tree with n leaves there is a toric degeneration $Cox(\mathcal{M}_{\mathbb{P}^1,\vec{p}}(SL_2(\mathbb{C}))) \Rightarrow \mathbb{C}[P_{\mathcal{T}}(1)]$

Definition:For Γ a trivalent graph of genus g with n marked points we define $P_{\Gamma}(L)$ to be the polytope given by non-negative integer weightings of the edges of Γ which satisfy the Quantum Clebsch-Gordon rules at each trinode with respect to level L.

Results of Sturmfels-Xu

This also gives a simultaneous Toric degeneration of the projective coordinate rings on $\mathcal{M}_{C,\vec{p}}(SL_2(\mathbb{C}))$.

Definition:For Γ a trivalent graph of genus g with n marked points, and $\vec{r}$ an $n-$vector of non-negative integers, we define $P_\Gamma(\vec{r}, L)$ to be the polytope given by non-negative integer weightings of the edges of Γ which satisfy the Quantum Clebsch-Gordon rules at each trinode with respect to level L and have the $i-$th leaf weight fixed to be r_i.

$$R_{\mathbb{P}^1,\vec{p}}(\vec{r}, L) \Rightarrow \mathbb{C}[P_{\mathcal{T}}(\vec{r}, L)]$$

Degeneration of $V_{C,\vec{p}}(G)$

Step 1) Turn the flat vector bundle theorem of Tsuchiya, Ueno, Yamada into a statement about algebras.

The quasi-coherant sheaf $V(G) = \bigoplus_{\vec{\lambda},L} V(\vec{\lambda}, L)$ has the structure of a multigraded flat sheaf of algebras over $\bar{\mathcal{M}}_{g,n}$. The fiber of this sheaf over a smooth $(C, \vec{p})$ is $Cox(\mathcal{M}_{C,\vec{p}}(G))$

-Note that this gives a multigraded algebra $V_{C,\vec{p}}(G)$ at $(C, \vec{p}) \in \bar{\mathcal{M}}_{g,n}$ non-smooth.

Degeneration of $V_{C,\vec{p}}(G)$

Step 2) Turn the factorization rules into a statement about algebras.

$$V_{C_\Gamma,\vec{p}}(\vec{\lambda}, L) \cong \bigoplus_{\alpha_j \in \Delta_L} V_{C',\vec{q}_1,\vec{q}_2,\vec{p}}(\vec{\alpha}, \vec{\alpha}^*, \vec{\lambda}, L)$$

$$V_{C_\Gamma,\vec{p}}(G) = \bigoplus_{\vec{\lambda},\vec{\alpha},L} V_{C',\vec{q}_1,\vec{q}_2,\vec{p}}(\vec{\alpha}, \vec{\alpha}^*, \vec{\lambda}, L)$$

$$\mathcal{R}_{\leq \vec{\beta},\vec{\gamma}} = \bigoplus_{\vec{\lambda} \leq \vec{\beta}, \vec{\alpha} \leq \vec{\gamma}, L} V_{C',\vec{q}_1,\vec{q}_2,\vec{p}}(\vec{\alpha}, \vec{\alpha}^*, \vec{\lambda}, L)$$

$$gr[V_{C_\Gamma,\vec{p}}(G)] = \bigoplus \mathcal{R}_{\leq \vec{\beta},\vec{\gamma}} / \mathcal{R}_{< \vec{\beta},\vec{\gamma}}$$

Degeneration of $V_{C,\vec{p}}(G)$

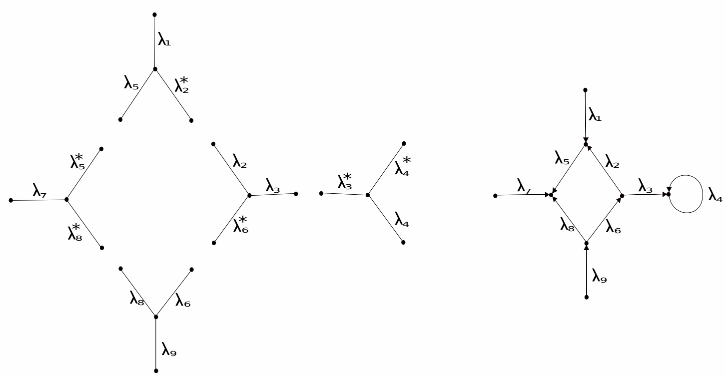
[M]: For $(C_\Gamma, \vec{p}) \in \bar{\mathcal{M}}_{g,n}$ the singular curve with dual graph Γ, there is a flat degeneration, $V_{C_\Gamma,\vec{p}}(G) \Rightarrow [\bigotimes_{v \in V(\Gamma)} V_{0,3}(G)]^{T_\Gamma}$.

Here $[\bigotimes_{v \in V(\Gamma)} V_{0,3}(G)]^{T_\Gamma}$ is the ring of invariants in $\bigotimes_{v \in V(\Gamma)} V_{0,3}(G)$ with respect to the action of a certain torus.

Degeneration of $V_{C,\vec{p}}(G)$

-Associate a copy of $\bigotimes_{v\in V(\Gamma)} V_{0,3}(G)$ to each $v \in V(\Gamma)$.

-Take the sub-ring where weight data from two different vertices v_1, v_2, which share a common edge e is dual, as in the factorization rules.



Degeneration of $V_{C,\vec{p}}(G)$

[M]: For every trivalent graph Γ with n leaves, there is a flat degeneration

$$Cox(\mathcal{M}_{C,\vec{p}}(G)) \Rightarrow [\bigotimes_{v,\in V(\Gamma)} Cox(\mathcal{M}_{0,3}(G))]^{T_\Gamma}$$

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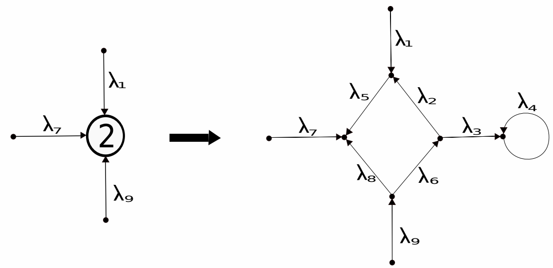
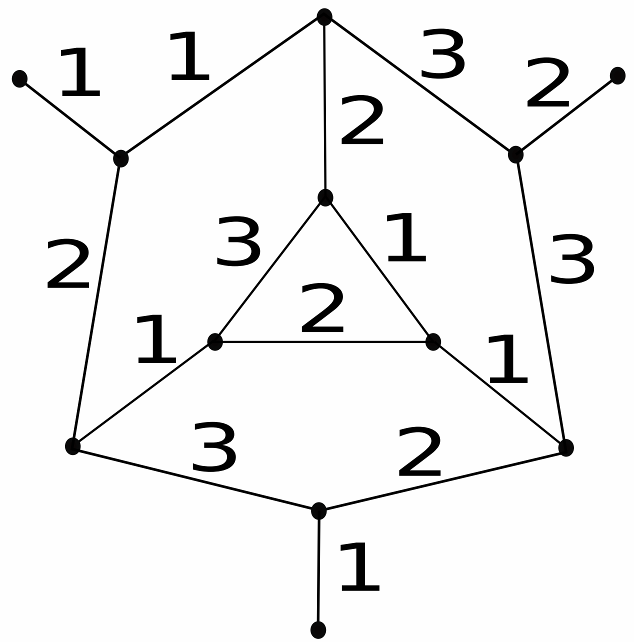


Results: $SL_2(\mathbb{C})$ case

Reduces the study of $Cox(\mathcal{M}_{C,\vec{p}}(SL_2(\mathbb{C})))$ to the study of these objects:



Results: $SL_2(\mathbb{C})$ case

Corollary [M]: There are flat degenerations

$$Cox(\mathcal{M}_{C,\vec{p}}(SL_2(\mathbb{C}))) \Rightarrow \mathbb{C}[P_\Gamma(1)]$$

$$R_{C,\vec{p}}(\vec{r}, L) \Rightarrow \mathbb{C}[P_\Gamma(\vec{r}, L)]$$

This is a simultaneous generalization of Sturmfels-Xu, and also work of Abe, which utilized similar degenerations in the $SL_2(\mathbb{C})$ case to prove that $Cox(\mathcal{M}_C(SL_2(\mathbb{C})))$ is generated in degrees $1, 2$.

Results: Degeneration from the $k-$Pieri rule

We consider the line bundle $\mathcal{L}(\vec{r}\omega_1, L)$ on $\mathcal{M}_{\mathbb{P}^1,\vec{p}}(SL_m(\mathbb{C}))$ with dominant weights $r_i\omega_1$, all multiples of the first fundamental weight.

Corollary [M]: There is a flat degeneration

$$R_{C,\vec{p}}(\vec{r}\omega_1, L) \Rightarrow \mathbb{C}[GT(\vec{r}, L)].$$

These polytopes and the $SL_2(\mathbb{C})$ polytopes have special properties, which we discuss next talk.

Tensors and Conformal Blocks

-How to generalize when 3$-$point spaces are not multiplicity free?

[Tsuchiya, Ueno, Yamada]: The space $V_{0,3}(\lambda, \eta, \mu, L)$ can be identified with a subspace of $[V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^{\mathfrak{g}}$

Tensors and Conformal Blocks

Let $\theta(sl_2(\mathbb{C})) \subset \mathfrak{g}$ be the copy of $sl_2(\mathbb{C})$ corresponding to the longest root. We branch each $V(\lambda)$ along this sub-algebra.

$$V(\lambda^*) = \bigoplus_i W_{\lambda,i}$$

Let $W_L(\lambda, \eta, \mu) = \bigoplus_{i+j+k \leq 2L} W_{\lambda,i} \otimes W_{\eta,j} \otimes W_{\mu,k}$.

$$V_{0,3}(\lambda, \eta, \mu, L) = W_L(\lambda, \eta, \mu) \cap [V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^{\mathfrak{g}}$$

Thankyou!