Combinatorial commutative algebra of conformal blocks

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Outline of Talks

Talk 1: Review combinatorics and commutative algebra attached to conformal blocks, introduce flat degenerations of coordinate rings of moduli of principal bundles.

Talk 2: The $SL_2(\mathbb{C})$ case; structure of conformal blocks polytopes; combinatorial commutative algebra of $SL_2(\mathbb{C})$ conformal blocks.

Talk 3: The $SL_3(\mathbb{C})$ case; combinatorics of tensors; relationship between conformal blocks and mathematical biology.
Notation (everything over $\mathbb{C}$)

g: Simple Lie algebra.

$G$: Simple, simply-connected algebraic group ($\text{Lie}(G) = g$).

$\lambda$: dominant weight of $g, G$.

$\Delta$: a Weyl chamber of $g, G$.

$(C, \bar{p}) \in \bar{\mathcal{M}}_{g,n}$: stable $n$-marked genus $g$ curve.
Notation

\( \Delta_L \): The \( L \)-restricted part of \( \Delta \).

\( V_{C,p}(\vec{\lambda}, L) \): conformal blocks with weight data \( \vec{\lambda}, \lambda_i \in \Delta \), and level \( L \).

\( V_{g,n}(\vec{\lambda}, L) \): the dimension of \( V_{C,p}(\vec{\lambda}, L) \).

\( M_{C,p}(G) \): moduli of quasi-parabolic principal \( G \)-bundles. The parabolic structure at each marked point will come from a Borel subgroup \( B \subset G \) unless indicated otherwise.

\( L(\vec{\lambda}, L) \): line bundle on \( M_{C,p}(G) \) with weight data \( \vec{\lambda} \), level \( L \).
Let $\mathcal{X}(B)$ be the character group of the Borel $B \subset G$.

[Beauville, Laszlo, Sorger]

$$\text{Pic}(\mathcal{M}_{C,p}(G)) = \mathcal{X}(B)^n \times \mathbb{Z}$$
There is a natural isomorphism

$$H^0(\mathcal{M}_{C,\bar{p}}(G), \mathcal{L}(\lambda, L)) = V_{C,\bar{p}}(\lambda, L).$$

**Definition:**

$$R_{C,\bar{p}}(\lambda, L) = \bigoplus_{N \geq 0} V_{C,\bar{p}}(N\lambda, NL)$$
Consequence:

The vector space $V_{C,p}(G) = \bigoplus_{\lambda,L} V_{C,p}(\tilde{\lambda}, L)$ can be given the structure of a multigraded commutative ring, isomorphic to $Cox(M_{C,p}(G))$. 
Innocent questions

- How do we generate all conformal blocks?
- What relations must hold between them?
- How do we count the dimensions of spaces of conformal blocks?
Innocent questions [Commutative algebra]

- What multigraded components generate $V_{C,\bar{p}}(G)$?
- What relations hold among these generators?
- What is the multigraded Hilbert function of $V_{C,\bar{p}}(G)$?
The moduli stack $\tilde{M}_{g,n}$

Example: $\tilde{M}_{2,0}$

Let $(C_\Gamma, \bar{p}) \in \tilde{M}_{g,n}$ be the curve associated to the trivalent graph $\Gamma$. 
The dimension of $V_{C,\vec{p}}(\vec{\lambda}, L)$ depends only on the data $(\vec{\lambda}, L)$, the genus $g$, and the number of points $n$.

**Theorem** [Tsuchiya, Ueno, Yamada]:
For every $\vec{\lambda}, L$ there is a vector bundle $V(\vec{\lambda}, L)$ on $\bar{\mathcal{M}}_{g,n}$, with fiber over $(C, \vec{p})$ equal to $V_{C,\vec{p}}(\vec{\lambda}, L)$.

- Note that $(C, \vec{p})$ need not be smooth.
Geometry for counting rules

For \((C, \vec{p})\) a stable curve with a doubled point \(q \in C\), there is a normalized stable marked curve \((C', q_1, q_2, \vec{p})\).
Conformal Blocks: Factorization

For a marked curve \((C, \vec{p})\) with a doubled point \(q\),

[Tsuchiya, Ueno, Yamada; Faltings]

\[
V_{C, \vec{p}}(\vec{\lambda}, L) \cong \bigoplus_{\alpha \in \Delta_L} V_{C', q_1, q_2, \vec{p}}(\alpha, \alpha^*, \vec{\lambda}, L)
\]
Conformal Blocks: Factorization

The flatness of $V(\lambda, L)$ and the factorization rules allow us to pass from general curves to 3–marked genus 0 curves.

\[
\begin{align*}
\begin{bmatrix}
\lambda_1 & \lambda_2 \\
\lambda_3 & \lambda_4
\end{bmatrix} &= \\
\begin{bmatrix}
\lambda_1 \\
\lambda_3
& \lambda_4
\end{bmatrix} & = & \sum_{\alpha \in \Delta_L} \begin{bmatrix}
\lambda_1 \\
\lambda_3 & \lambda_4
\end{bmatrix} & \otimes & \begin{bmatrix}
\lambda_2 \\
\lambda_4
\end{bmatrix}
\end{align*}
\]
Conformal Blocks: Factorization

\[
\begin{align*}
\lambda_1 & \quad \lambda_2 \\
\lambda_5 & \quad \lambda_6 \\
\lambda_7 & \quad \lambda_8 \\
\lambda_9 & \quad \lambda_3 \\
\end{align*}
\]
Some terminology:

A rule for counting things: A formula or other method for evaluating the number of things.

A combinatorial rule for counting things: An explicitly integer formula (may involve some minus signs).

A positive counting rule for counting things: An explicitly non-negative integer formula.

A polyhedral counting rule for counting things: Bijection of things with the lattice points in a polytope.
An (by no means complete) overview of counting rules for conformal blocks

A counting rule (The Verlinde Formula):

\[ \mathbb{V}_{g,n}(\vec{\lambda},L) = |T_L|^{g-1} \sum_{\mu \in \Delta_L} Tr_{\vec{\lambda}}(exp(2\pi i \frac{(\mu + \rho)}{L + h^\vee})) \prod_\alpha |2\sin(\pi \frac{(\alpha|\mu + \rho)}{L + h^\vee})|^{2-2g} \]

-Reduce to the \( g = 0, n = 3 \) point case, use character theory of the Fusion algebra.

\[ X_{\lambda,L} \boxtimes X_{\eta,L} = \sum_{\alpha \in \Delta_L} \mathbb{V}_{0,3}(\lambda, \eta, \alpha^*, L) X_{\alpha,L} \]
A combinatorial rule (The Kac-Walton Formula)

$\hat{W} –$ the affine Weyl group.

$$\mathbb{V}_{0,3}(\lambda, \eta, \mu, L) = \sum_{w \in \hat{W}, w \circ \tau = \mu} \epsilon(w) c_{\lambda, \eta, \tau}$$

$c_{\lambda, \eta, \tau} = \text{dim}( [V(\lambda) \otimes V(\eta) \otimes V(\tau)]^g)$

$w \circ \tau = w(\tau + \rho) - \rho$

$\rho = \sum \omega_i$
Korff and Stroppel have found a combinatorial “quantum Littlewood-Richardson rule” by presenting the fusion algebra in type $A$ using an affine version of the Plactic algebra. The resulting rule is distinct from the Kac-Walton formula.

Morse and Schilling have found combinatorial rules for conformal blocks with special kinds of weight data, expressed as “cylindrical tableaux.” This involves Kostka matrices, and gives a positive counting rule.
Buch, Kresch, Tamvakis have formulated a positive, conjectural counting rule by linking the dimension of the space of conformal blocks in type $A$ to a classical intersection number on a two-step flag variety.

This leads to a formulation in terms of A. Knutson’s puzzles.
\( \mathbb{L} \subset \mathbb{R}^n \) a lattice, \( A \in M_{m,n}(\mathbb{R}) \), a rational matrix, \( b \in \mathbb{R}^m \)

\[
P_{A,b} = \{ v | Av \leq b \}
\]

Polytope: a compact solution set to a set of inequalities.

\( P_{A,b} \cap \mathbb{L} \) - the lattice points in the polytope.
Polyhedral counting rules: $SL_2(\mathbb{C})$

In the case $G = SL_2(\mathbb{C})$, the spaces $V_{0,3}(r_1, r_2, r_3, L)$ are dimension 1 or 0.

Proposition [Quantum Clebsch-Gordon rule]:
The dimension of $V_{0,3}(r_1, r_2, r_3, L)$ is either 1 or 0. It is dimension 1 if and only if $r_1 + r_2 + r_3$ is even, $\leq 2L$, and $r_1, r_2, r_3$ are the side-lengths of a triangle.
Polyhedral counting rules: $SL_3(\mathbb{C})$

$\mathcal{V}_{0,3}(\lambda, \eta, \mu, L) = \text{The number of diagrams below with}$

- all $\geq 0$ integer entries,
- $(a_1 + a_2, a_3 + a_4) = \lambda$, $(b_1 + b_2, b_3 + b_4) = \eta$, $(c_1 + c_2, c_3 + c_4) = \mu$,
- $a_2 + a_3 = c_2 + b_3$, $b_2 + b_3 = a_2 + c_3$, $c_2 + c_3 = b_2 + a_3$,
- $a_1 + b_1 + c_1 + ... \leq L$
Polyhedral counting rules: $SL_3(\mathbb{C})$

$$\forall_{0,3}(\lambda, \mu, \eta, L) = L - \max\{\lambda_1 + \lambda_2, \mu_1 + \mu_2, \eta_1 + \eta_2, L_1, L_2\} + 1$$

$$L_1 = \frac{1}{3}(2(\lambda_1 + \mu_1 + \eta_1) + \lambda_2 + \mu_2 + \eta_2) - \min\{\lambda_1, \mu_1, \eta_1\}$$

$$L_2 = \frac{1}{3}(2(\lambda_2 + \mu_2 + \eta_2) + \lambda_1 + \mu_1 + \eta_1) - \min\{\lambda_2, \mu_2, \eta_2\}.$$

-Senechal, Mathieu, Kirillov, and Walton; [M]
Polyhedral counting rules

A polyhedral counting rule in the $g = 0, n = 3$ case yields a polyhedral counting rule in all cases via toric fiber products.

\[
\pi_1 : P \to D, \pi_2 : Q \to D,
\]

\[
P \times_D Q = \{ (v, w) \mid v \in P, w \in Q, \pi_1(v) = \pi_2(w) \}\]
Polyhedral counting rules

Given a polyhedral rule to evaluate $V_{0,3}$, there is a distinct polyhedral counting rule for evaluating $V_{g,n}$ for each special curve $(C_{\Gamma}, \vec{p}) \in \bar{\mathcal{M}}_{g,n}$.
Polyhedral Counting rules: $k$–Pieri rule

For $G = SL_m(\mathbb{C})$, $\lambda_i = r_i \omega_1$, the dimension of $V_{p,1,\bar{p}}(\vec{r}\omega_1, L)$ is the number of lattice points in $GT(\vec{r}, L)$, given by non-negative integer fillings of the following interlacing diagram.

This is an application of the "$k$–Pieri rule."
Combinatorial commutative algebra

Each polytope $P \subset (\mathbb{R}^n, \mathbb{L})$ has an associated affine semigroup algebra $\mathbb{C}[P]$.

This is constructed by considering the lattice points in the Minkowski sums of $P$.

\[
k \circ P = \{v_1 + \ldots + v_k | v_i \in P\}
\]

\[
\mathbb{C}[P]_k = \{v | v \in (k \circ P) \cap \mathbb{L}\}
\]

\[
\mathbb{C}[P]_k \times \mathbb{C}[P]_\ell \to \mathbb{C}[P]_{k+\ell}
\]
Combinatorial commutative algebra

These algebras are "nice" (or at least nicer than your average commutative algebra):

- always finitely generated (Gordon’s Lemma),
- always cut out by binomial relations,
- always Cohen-Macaulay,
- software: 4ti2, polymake, Macaulay 2.
Combinatorial commutative algebra of conformal blocks

Each of the previous polyhedral counting rules comes with a natural algebraic structure coming from affine semigroup multiplication.

Question: What does this structure have to do with $V_{C,\vec{p}}(G)$?
Each of the previous polyhedral counting rules comes with a natural algebraic structure coming from affine semigroup multiplication.

Question: What does this structure have to do with $V_{C,\vec{p}}(G)$?

Strategy: Find a toric degeneration of $V_{C,\vec{p}}(G)$.
- give polyhedral counting rules
- use to study algebraic structure.
Results of Sturmfels-Xu

[Sturmfels, Xu]: For each $\mathcal{T}$ a trivalent tree with $n$ leaves there is a toric degeneration $Cox(\mathcal{M}_{\mathbb{P}^1,\vec{p}}(SL_2(\mathbb{C}))) \Rightarrow \mathbb{C}[P_{\mathcal{T}}(1)]$

Definition: For $\Gamma$ a trivalent graph of genus $g$ with $n$ marked points we define $P_{\Gamma}(L)$ to be the polytope given by non-negative integer weightings of the edges of $\Gamma$ which satisfy the Quantum Clebsch-Gordon rules at each trinode with respect to level $L$. 
Results of Sturmfels-Xu

This also gives a simultaneous Toric degeneration of the projective coordinate rings on \( \mathcal{M}_{C,\vec{p}}(SL_2(\mathbb{C})) \).

Definition: For \( \Gamma \) a trivalent graph of genus \( g \) with \( n \) marked points, and \( \vec{r} \) an \( n \)–vector of non-negative integers, we define \( P_\Gamma(\vec{r}, L) \) to be the polytope given by non-negative integer weightings of the edges of \( \Gamma \) which satisfy the Quantum Clebsch-Gordon rules at each trinode with respect to level \( L \) and have the \( i \)–th leaf weight fixed to be \( r_i \).

\[
R_{\mathbb{P}^1, \vec{p}}(\vec{r}, L) \Rightarrow \mathbb{C}[P_\Gamma(\vec{r}, L)]
\]
Degeneration of $V_{C,\vec{p}}(G)$

Step 1) Turn the flat vector bundle theorem of Tsuchiya, Ueno, Yamada into a statement about algebras.

The quasi-coherant sheaf $V(G) = \bigoplus_{\lambda,L} V(\tilde{\lambda}, L)$ has the structure of a multigraded flat sheaf of algebras over $\bar{\mathcal{M}}_{g,n}$. The fiber of this sheaf over a smooth $(C, \vec{p})$ is $\text{Cox}(\mathcal{M}_{C,\vec{p}}(G))$.

-Note that this gives a multigraded algebra $V_{C,\vec{p}}(G)$ at $(C, \vec{p}) \in \bar{\mathcal{M}}_{g,n}$ non-smooth.
Degeneration of $V_{C,\vec{p}}(G)$

Step 2) Turn the factorization rules into a statement about algebras.

\[
V_{C,\vec{p}}(\tilde{\lambda}, L) \cong \bigoplus_{\alpha_j \in \Delta_L} V_{C',\vec{q}_1,\vec{q}_2,\vec{p}}(\tilde{\alpha}, \tilde{\alpha}^*, \tilde{\lambda}, L)
\]

\[
V_{C,\vec{p}}(G) = \bigoplus_{\tilde{\lambda}, \tilde{\alpha}, L} V_{C',\vec{q}_1,\vec{q}_2,\vec{p}}(\tilde{\alpha}, \tilde{\alpha}^*, \tilde{\lambda}, L)
\]

\[
\mathcal{R}_{\leq \tilde{\beta}, \tilde{\gamma}} = \bigoplus_{\tilde{\lambda} \leq \tilde{\beta}, \tilde{\alpha} \leq \tilde{\gamma}, L} V_{C',\vec{q}_1,\vec{q}_2,\vec{p}}(\tilde{\alpha}, \tilde{\alpha}^*, \tilde{\lambda}, L)
\]

\[
gr[V_{C,\vec{p}}(G)] = \bigoplus \mathcal{R}_{\leq \tilde{\beta}, \tilde{\gamma}} / \mathcal{R}_{< \tilde{\beta}, \tilde{\gamma}}
\]
Degeneration of $V_{C,\vec{p}}(G)$

[1]: For $(C_{\Gamma}, \vec{p}) \in \bar{\mathcal{M}}_{g,n}$ the singular curve with dual graph $\Gamma$, there is a flat degeneration, $V_{C_{\Gamma},\vec{p}}(G) \Rightarrow [\bigotimes_{v \in V(\Gamma)} V_{0,3}(G)]^{T_{\Gamma}}$.

Here $[\bigotimes_{v \in V(\Gamma)} V_{0,3}(G)]^{T_{\Gamma}}$ is the ring of invariants in $\bigotimes_{v \in V(\Gamma)} V_{0,3}(G)$ with respect to the action of a certain torus.
Degeneration of $V_{C,\vec{p}}(G)$

- Associate a copy of $\bigotimes_{v \in V(\Gamma)} V_{0,3}(G)$ to each $v \in V(\Gamma)$.
- Take the sub-ring where weight data from two different vertices $v_1, v_2$, which share a common edge $e$ is dual, as in the factorization rules.
Degeneration of $V_{C,\tilde{p}}(G)$

[M]: For every trivalent graph $\Gamma$ with $n$ leaves, there is a flat degeneration

$$\text{Cox}(\mathcal{M}_{C,\tilde{p}}(G)) \Rightarrow \left[ \bigotimes_{v \in V(\Gamma)} \text{Cox}(\mathcal{M}_{0,3}(G)) \right]^{T_{\Gamma}}$$
Results: $SL_2(\mathbb{C})$ case

Reduces the study of $\text{Cox}(\mathcal{M}_{C,p}(SL_2(\mathbb{C})))$ to the study of these objects:
Results: $SL_2(\mathbb{C})$ case

Corollary [M]: There are flat degenerations

$$\text{Cox}(\mathcal{M}_{C,p}(SL_2(\mathbb{C}))) \Rightarrow \mathbb{C}[P_{\Gamma}(1)]$$

$$R_{C,p}(\vec{r}, L) \Rightarrow \mathbb{C}[P_{\Gamma}(\vec{r}, L)]$$

This is a simultaneous generalization of Sturmfels-Xu, and also work of Abe, which utilized similar degenerations in the $SL_2(\mathbb{C})$ case to prove that $\text{Cox}(\mathcal{M}_C(SL_2(\mathbb{C})))$ is generated in degrees 1, 2.
Results: Degeneration from the $k$–Pieri rule

We consider the line bundle $\mathcal{L}(\vec{r}\omega_1, L)$ on $\mathcal{M}_{\mathbb{P}^1, \vec{p}}(SL_m(\mathbb{C}))$ with dominant weights $r_i\omega_1$, all multiples of the first fundamental weight.

**Corollary [M]:** There is a flat degeneration

$$R_{\mathbb{C}, \vec{p}}(\vec{r}\omega_1, L) \Rightarrow \mathbb{C}[GT(\vec{r}, L)].$$

These polytopes and the $SL_2(\mathbb{C})$ polytopes have special properties, which we discuss next talk.
- How to generalize when 3–point spaces are not multiplicity free?

[Tsuchiya, Ueno, Yamada]: The space $V_{0,3}(\lambda, \eta, \mu, L)$ can be identified with a subspace of $[V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^g$
Tensors and Conformal Blocks

Let $\theta(sl_2(\mathbb{C})) \subset g$ be the copy of $sl_2(\mathbb{C})$ corresponding to the longest root. We branch each $V(\lambda)$ along this sub-algebra.

$$V(\lambda^*) = \bigoplus_i W_{\lambda,i}$$

Let $W_L(\lambda, \eta, \mu) = \bigoplus_{i+j+k \leq 2L} W_{\lambda,i} \otimes W_{\eta,j} \otimes W_{\mu,k}$.

$$V_{0,3}(\lambda, \eta, \mu, L) = W_L(\lambda, \eta, \mu) \cap [V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^g$$
Thankyou!